Math 5260: Problem Set 1 (Philosophy), Due Thursday Sept 5

For the first three problems, recall that a function $f : X \to Y$ is one-to-one if $x_0 \neq x_1$ implies that $f(x_0) \neq f(x_1)$. One way to show that a given f is one-to-one is to apply this definition directly. That is, assume that $x_0 \neq x_1$ are distinct elements of X and show that $f(x_0) \neq f(x_1)$ are distinct elements of Y.

The first problem uses the following definition. Let $(A, \leq_A)$ and $(B, \leq_B)$ be linear orders. A function $f : A \to B$ is *strictly increasing* if $a_0 <_A a_1$ implies that $f(a_0) <_B f(a_1)$.

Problem 1. Let $(A, \leq_A)$ and $(B, \leq_B)$ be linear orders and let $f : A \to B$ be strictly increasing. Prove that f is one-to-one.

Hint. Try verifying that f is one-to-one directly from the definition. Assume that $a_0 \neq a_1$ are district elements of A. You need to show that $f(a_0) \neq f(a_1)$ are distinct elements of B. Think about the fact that a_0 and a_1 are district elements of the linear order $(A, \leq_A)$. What can you say about their order relationship? How does that help you use the fact that f is strictly increasing to conclude that $f(a_0) \neq f(a_1)$?

For the second problem, consider the relationship between $\mathcal{P}(A)$ and 2^A. In class, we indicated that you could show $|\mathcal{P}(A)| = |2^A|$ by considering the function $\Delta : \mathcal{P}(A) \to 2^A$ given by $\Delta(Y) = \chi_Y$. The input Y is a subset $Y \subseteq A$ and the output $\chi_Y : A \to \{0, 1\}$ is called the characteristic function of Y and is defined by

$$\chi_Y(a) = \begin{cases} 0 & \text{if } a \notin Y \\ 1 & \text{if } a \in Y \end{cases}$$

Problem 2. Let A be a set and let $\Delta : \mathcal{P}(A) \to 2^A$ be the function $\Delta(Y) = \chi_Y$. Prove that Δ is one-to-one.

Hint. Again, I would do this problem directly. Assume that $Y_0 \neq Y_1$ are district subsets of A. Show that $\Delta(Y_0) \neq \Delta(Y_1)$. To show that $\Delta(Y_0) \neq \Delta(Y_1)$, you need to show that $\chi_{Y_0} \neq \chi_{Y_1}$ as functions. This means that they must differ on some input. In other words, you need to show that there is some $a \in A$ such that $\chi_{Y_0}(a) \neq \chi_{Y_1}(a)$. Think about what it means for $Y_0 \neq Y_1$. That is, what must happen for two sets to be different? How does that help you find an appropriate a for which $\chi_{Y_0}(a) \neq \chi_{Y_1}(a)$?

Another way to show that $f : X \to Y$ is one-to-one is to consider the contrapositive of the definition of one-to-one. That is, f is one-to-one if $f(x_0) = f(x_1)$ implies that $x_0 = x_1$. Thinking about this form of the definition, you would show f is one-to-one by assuming that $f(x_0) = f(x_1)$ and then showing that $x_0 = x_1$.

Problem 3. Show that if $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

Hint. For this problem, you can fix one-to-one functions $f : A \to B$ and $g : B \to C$. You need to show there is a one-to-one function $h : A \to C$. By comments in class, you should set h to be the composition $h = g \circ f$, i.e. for $a \in A$, $h(a) = (g \circ f)(a) = g(f(a))$. That is, first apply f to a to get $f(a) \in B$. Then apply g to $f(a)$ to get $g(f(a)) \in C$. To show h is one-to-one, assume that $h(a_0) = h(a_1)$. This means $g(f(a_0)) = g(f(a_1))$. Since g is one-to-one, what can you say about the relationship between $f(a_0)$ and $f(a_1)$? Next use the fact that f is one-to-one. What can you conclude about the relationship between a_0 and a_1?

The next problem is an example of showing a function is onto. We consider $\Delta : \mathcal{P}(A) \to 2^A$ given by $\Delta(Y) = \chi_Y$. To show this function is onto, we start with an element $g \in 2^A$, i.e. with a function $g : A \to \{0, 1\}$. We need to show that there is some $Y \in \mathcal{P}(A)$ such that $\Delta(Y) = g$. That is, we need to show that there is a subset $Y \subseteq A$ such that $g = \chi_Y$.

Problem 4. Prove that $\Delta : \mathcal{P}(A) \to 2^A$ given by $\Delta(Y) = \chi_Y$ is onto.

Hint. Fix $g \in 2^A$, i.e. $g : A \to \{0, 1\}$. Describe a subset $Y \subseteq A$ such that $g = \chi_Y$.

Problem 5. Prove that $|\mathbb{N}| = |\mathbb{Z}|$.

Problem 6. Let A be a set. Prove that $|A| < |\mathcal{P}(A)|$.

Hint. This problem has two parts. First, you need to show that $|A| \leq |\mathcal{P}(A)|$ by giving a one-to-one function $f : A \to \mathcal{P}(A)$. Second, you need to show that $|A| \neq |\mathcal{P}(A)|$ by proving there is no bijection $g : A \to \mathcal{P}(A)$. Fix $g : A \to \mathcal{P}(A)$. To show g is not a bijection, it suffices to show g is not onto. Prove the subset $Y \subseteq A$ given by

$$Y = \{a \in A \mid a \notin g(a)\}$$

is not in the range of g. For a contradiction, assume that there is a $b \in A$ such that $g(b) = Y$. Is $b \in Y$? Both possible answers should lead you to a contradiction.

Problem 7. Prove that a countable union of countable sets is countable. That is, let A_i, for $i \in \mathbb{N}$, be a family of sets such that each A_i is countable. Prove that $\bigcup_{i \in \mathbb{N}} A_i$ is countable.

Hint. You can fix bijections $g_i : A_i \to \mathbb{N}$ for each $i \in \mathbb{N}$. Let $B = \bigcup_{i \in \mathbb{N}} A_i$. You need to show $|\mathbb{N}| \leq |B|$ and $|B| \leq |\mathbb{N}|$. For the second inequality, it suffices to show $|B| \leq |\mathbb{N} \times \mathbb{N}|$. The temptation is to try to define a one-to-one function $f : B \to \mathbb{N} \times \mathbb{N}$ by specifying $f(b)$ as follows: Since $b \in B$, we know $b \in A_i$ for some i, so let $f(b) = \langle i, g_i(b) \rangle$. The difficulty is that the sets A_i need not be disjoint. That is, if $b \in B$, then you know $b \in A_i$ for some i. But, in fact, you could have $b \in A_i$ for many indices i.

Problem 8. Prove that the order defined in Example 3.11 of the notes is a well order. (*Hint.* Look at Example 3.8.)

Problem 9. Prove that the order defined in Example 3.12 of the notes is a well order. (*Hint.* Look at Example 3.9.)

Problem 10. Prove that $(\mathbb{Z}, \leq_Z) \ncong (\mathbb{Q}, \leq_{\mathbb{Q}})$.

Hint. Suppose for a contradiction that $f : \mathbb{Z} \to \mathbb{Q}$ is an isomorphism. Since $0 <_{\mathbb{Z}} 1$, you know $f(0) <_{\mathbb{Q}} f(1)$. Consider the element $q = (f(0) + f(1))/2 \in \mathbb{Q}$. Where does it sit in the ordering on $\mathbb{Q}$? Show that there cannot be an element $z \in \mathbb{Z}$ such that $f(z) = q$.

If you want to try a challenging problem, here one more. This doesn't use anything mathematical beyond careful manipulations of functions, but the argument is more involved than anything above. You don't have to hand this one in, but at some point, it is a good exercise to try to work through. The proof of the Schroeder-Bernstein Theorem outlined below does not us AC.

Bonus Problem. Prove the Schroeder-Bernstein Theorem by showing that $|A| = |B|$ if and only if $|A| \leq |B|$ and $|B| \leq |A|$. (See back side for hint.)

Hint. One direction is trivial. For the nontrivial direction, assume that $|A| \leq |B|$ and $|B| \leq |A|$ and prove that $|A| = |B|$. Fix one-to-one functions $f : A \to B$ and $g : B \to A$. We need to define a bijection $h : A \to B$. Define decreasing sequences of subsets of A and B indexed by $\mathbb{N}$

$$A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$$
$$B = B_0 \supseteq B_1 \supseteq B_2 \supseteq A_3 \supseteq \cdots$$

by induction. For the base case, set $A_0 = A$ and $B_0 = B$. For the induction step, set $A_{n+1} = g[B_n]$ and $B_{n+1} = f[A_n]$.

Step 1. Consider A_0, A_1, A_2, B_0, B_1 and B_2.

(a) Prove that f gives a bijection between $A_0 \setminus A_1$ and $B_1 \setminus B_2$.

(b) Analogously, prove that g gives a bijection between $B_0 \setminus B_1$ and $A_1 \setminus A_2$.

Step 2. Using Step 1, show that $h : A_0 \setminus A_2 \to B_0 \setminus B_2$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_0 \setminus A_1 \\ g^{-1}(x) & \text{if } x \in A_1 \setminus A_2 \end{cases}$$

is a bijection.

Step 3. Using essentially the same arguments, show that for any $n \in \mathbb{N}$:

(a) f gives a bijection between $A_{2n} \setminus A_{2n+1}$ and $B_{2n+1} \setminus B_{2n+2}$ and

(b) g gives a bijection between $B_{2n} \setminus B_{2n+1}$ and $A_{2n+1} \setminus A_{2n+2}$.

Step 4. Let $A_\infty = \bigcap_{n\in\mathbb{N}} A_n$ and $B_\infty = \bigcap_{n\in\mathbb{N}} B_n$. Prove that f gives a bijection between A_∞ and B_∞.

Step 5. Prove that $h : A \to B$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_{2n} \setminus A_{2n+1} \text{ for some } n \\ g^{-1}(x) & \text{if } x \in A_{2n+1} \setminus A_{2n+2} \text{ for some } n \\ f(x) & \text{if } x \in A_\infty \end{cases}$$

is a bijection.