Math 5260: Problem Set 1 (Math), Due Thursday Sept 5

Problem 1. Let A be a set. Prove that $|\mathcal{P}(A)| = |2^A|$ by showing that the function $\Delta : \mathcal{P}(A) \to 2^A$ given by $\Delta(Y) = \chi_Y$ is a bijection.

Problem 2. Let A be a set. Prove that $|A| < |\mathcal{P}(A)|$.

Hint. This problem has two parts. First, you need to show that $|A| \leq |\mathcal{P}(A)|$ by giving a one-to-one function $f : A \to \mathcal{P}(A)$. Second, you need to show that $|A| \neq |\mathcal{P}(A)|$ by proving there is no bijection $g : A \to \mathcal{P}(A)$. Fix $g : A \to \mathcal{P}(A)$. To show g is not onto, prove the subset $Y \subseteq A$ given by

$$Y = \{a \in A \mid a \notin g(a)\}$$

is not in the range of g. For a contradiction, assume $g(b) = Y$ for some $b \in A$. Is $b \in Y$?

Problem 3. Prove that a countable union of countable sets is countable. That is, let A_i, for $i \in \mathbb{N}$, be a family of sets such that each A_i is countable. Prove that $\bigcup_{i \in \mathbb{N}} A_i$ is countable.

Hint. You can fix bijections $g_i : A_i \to \mathbb{N}$ for each $i \in \mathbb{N}$. Let $B = \bigcup_{i \in \mathbb{N}} A_i$. You need to show $|\mathbb{N}| \leq |B|$ and $|B| \leq |\mathbb{N}|$. For the second inequality, it suffices to show $|B| \leq |\mathbb{N} \times \mathbb{N}|$. The temptation is to try to define a one-to-one function $f : B \to \mathbb{N} \times \mathbb{N}$ by specifying $f(b)$ as follows: Since $b \in B$, we know $b \in A_i$ for some i, so let $f(b) = \langle i, g_i(b) \rangle$. The difficulty is that the sets A_i need not be disjoint. That is, if $b \in B$, then you know $b \in A_i$ for some i. But, in fact, you could have $b \in A_i$ for many indices i.

Problem 4. Prove that the order defined in Example 3.11 of the notes is a well order. (*Hint.* Look at Example 3.8.)

Problem 5. Prove that the order defined in Example 3.12 of the notes is a well order. (*Hint.* Look at Example 3.9.)

For the last problem, the following notation is useful. Let $f : X \to Y$ be a function. For a subset $Z \subseteq X$, let

$$f[Z] = \{y \in Y \mid \exists z \in Z\, (f(z) = y)\}$$

That is, $f[Z] = \text{range}(f \upharpoonright Z)$.

Problem 6. Prove the Schroeder-Bernstein Theorem by showing that $|A| = |B|$ if and only if $|A| \leq |B|$ and $|B| \leq |A|$. (See back side for hint.)

Hint. One direction is trivial. For the nontrivial direction, assume that $|A| \leq |B|$ and $|B| \leq |A|$ and prove that $|A| = |B|$. Fix one-to-one functions $f : A \to B$ and $g : B \to A$. We need to define a bijection $h : A \to B$. Define decreasing sequences of subsets of A and B indexed by $\mathbb{N}$

$$\begin{aligned} A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \\ B = B_0 \supseteq B_1 \supseteq B_2 \supseteq A_3 \supseteq \cdots \end{aligned}$$

by induction. For the base case, set $A_0 = A$ and $B_0 = B$. For the induction step, set $A_{n+1} = g[B_n]$ and $B_{n+1} = f[A_n]$.

Step 1. Consider A_0, A_1, A_2, B_0, B_1 and B_2.

(a) Prove that f gives a bijection between $A_0 \setminus A_1$ and $B_1 \setminus B_2$.

(b) Analogously, prove that g gives a bijection between $B_0 \setminus B_1$ and $A_1 \setminus A_2$.

Step 2. Using Step 1, show that $h : A_0 \setminus A_2 \to B_0 \setminus B_2$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_0 \setminus A_1 \\ g^{-1}(x) & \text{if } x \in A_1 \setminus A_2 \end{cases}$$

is a bijection.

Step 3. Using essentially the same arguments, show that for any $n \in \mathbb{N}$:

(a) f gives a bijection between $A_{2n} \setminus A_{2n+1}$ and $B_{2n+1} \setminus B_{2n+2}$ and

(b) g gives a bijection between $B_{2n} \setminus B_{2n+1}$ and $A_{2n+1} \setminus A_{2n+2}$.

Step 4. Let $A_\infty = \bigcap_{n \in \mathbb{N}} A_n$ and $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$. Prove that f gives a bijection between A_∞ and B_∞.

Step 5. Prove that $h : A \to B$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_{2n} \setminus A_{2n+1} \text{ for some } n \\ g^{-1}(x) & \text{if } x \in A_{2n+1} \setminus A_{2n+2} \text{ for some } n \\ f(x) & \text{if } x \in A_\infty \end{cases}$$

is a bijection.