Math 5026: Homework 2, Due Friday February 28

Problem 1. Let A be an index set. Use the Recursion Theorem to show that $A \not\leq_m \overline{A}$.

Problem 2. Let B be an infinite c.e. set. Prove there is a strictly increasing computable function f such that range$(f) \subseteq B$ and $n < f(n)$ for all n.

A set A is called *immune* if it is infinite but doesn't contain an infinite c.e. set. Later in the course, we will find these sets quite useful.

Problem 3. Let A be a c.e. set such that $\overline{A}$ is immune. Prove that A is not computable. (This proof is very short.)

Problem 4. Let $M = \{x \mid \neg(\exists y < x)\,(\varphi_x = \varphi_y)\}$. That is, M consists of the least index for each partial computable function. Note that M is infinite because there are infinitely many different partial computable functions. Prove that M is immune.

For the next problem, recall that $A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$. A minor modification of one of the problems from Homework 1 shows that if $A \leq_T C$ and $B \leq_T C$, then $A \oplus B \leq_T C$. In Problem 5, you show that the related least upper bound notion for the Turing degrees is well defined.

Problem 5. Prove that if $A \equiv_T D$ and $B \equiv_T E$, then $A \oplus B \equiv_T D \oplus E$. (This proof is also very short.)

The last two problems give an example of a related operation on sets which does not correspond to a well defined notion on the Turing degrees. Let $\{A_y \mid y \in \omega\}$ be a family of sets indexed by ω. Define

$$\oplus_{y\in\omega} A_y = \{\langle x, y\rangle \mid x \in A_y\}$$

Problem 6 shows why $\oplus_{y\in\omega} A_y$ is called the *uniform upper bound of the indexed family* A_y.

Problem 6. Let C be a set and f be a computable function such that $A_y = \Phi^C_{f(y)}$ for all y. That is, $A_y \leq_T C$ for all y, and the computable function f gives the indices for these reductions uniformly. Prove that $\oplus_{y\in\omega} A_y \leq_T C$. (This proof is again very short.)

Problem 7. Give an example of two families of sets A_y, $y \in \omega$, and B_y, $y \in \omega$, such that $A_y \equiv_T B_y$ for all y, but $\oplus_{y\in\omega} A_y \not\equiv_T \oplus_{y\in\omega} B_y$.

Hints for Homework 2

Problem 1. Suppose that $A \leq_m \overline{A}$. Apply the Recursion Theorem to the function witnessing this reduction.

Problem 2. Use the fact that B contains an infinite computable set A, and you can assume without loss of generality that $0 \notin A$. (Since if $0 \in A$, then you can remove 0 from A and still have an infinite computable subset of B.)

Problem 3. Think about why $\overline{A}$ cannot be c.e. and why this suffices for the proof.

Problem 4. You already know M is infinite, so you only need to show M doesn't contain an infinite c.e. set. Suppose that M does contain an infinite c.e. set B. Use Problem 2 and the Recursion Theorem to help you.

Problem 6. You need to described an oracle computation Φ^C that on input $\langle x, y \rangle$ uses C to determine if $x \in A_y$. You can describe this computation procedure using the things you are given in the problem.

Problem 7. I think the simplest examples keep all the sets computable. Try letting $A_y = \emptyset$ for all y. As long as each B_y set is computable, you will have $A_y \equiv_T B_y$. So, it suffices to describe a sequence of computable sets B_y such that from $\oplus_{y \in \omega} B_y$, you can compute something non-computable. Note that while each B_y has to be individually computable, you do not need to construct the sequence of sets B_y uniformly.