Math 5026: Extra Credit for Homework 1, Due Friday February 21

Quite a few people ran into some difficulties using the s-m-n theorem in the index set reduction problems on the first homework. Next week, we will be doing more complicated examples of index set reductions, so it would be good to review this material and make sure your understanding is solid. Towards this goal, I wrote out a couple of examples in detail below. I will give extra credit for homework 1 to anyone who wants to prove the following four reductions: $\{e \mid \varphi_e(0)\downarrow\} \leq_m K$, $K \leq_m \text{Inf}$, $K \leq_m \text{Tot}$ and $K \leq_m \text{Cof}$.

Example. Let $L = \{e \mid \varphi_e(0)\downarrow\}$. Show that $K \leq_m L$.

The s-m-n theorem arguments almost always have the same form. The first step is to define an appropriate partial computable function. What makes the function appropriate depends on the reduction you are trying to show. Often it is useful to work backwards from the end of the argument to see what would be helpful. An appropriate partial computable function is often a function of more than one variable that is defined in cases with a c.e. condition in one case (which gives an output if the c.e. condition holds) and the other case diverges (if the c.e. condition doesn't hold).

In this example, we consider the function

$$\psi(x,y) = \begin{cases} 0 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Since this function is partial computable, it has an index, which we will call e. That is, $\psi(x,y) = \varphi_e(x,y)$, so

$$\varphi_e(x,y) = \begin{cases} 0 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

The second step is to use the s-m-n theorem to push one of the variables down into the index. By the s-m-n theorem, we have $\varphi_e(x,y) = \varphi_{s_1^1(e,x)}(y)$, where s_1^1 is a 1-1 computable function (and, in particular, is total). Let $f(x) = s_1^1(e,x)$. Then f is a 1-1 computable function (and again, in particular, is total) and

$$\varphi_{f(x)}(y) = \begin{cases} 0 & \text{if } \varphi_x(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

The third step is to show that this function f gives the required reduction. In this case, we need to show that $x \in K \Leftrightarrow f(x) \in L$. Often, it is helpful to split showing this equivalence into showing the two implications separately. For the first direction,

$$x \in K \Rightarrow \forall y(\psi(x,y) = 0) \Rightarrow \forall y(\varphi_{f(x)}(y) = 0) \Rightarrow \varphi_{f(x)}(0)\downarrow \Rightarrow f(x) \in L$$

For the other direction,

$$x \notin K \Rightarrow \forall y(\psi(x,y)\uparrow) \Rightarrow \forall y(\varphi_{f(x)}(y)\uparrow) \Rightarrow \varphi_{f(x)}(0)\uparrow \Rightarrow f(x) \notin L$$

Example. Show that $\overline{K} \leq_m \text{Fin}$.

To find an appropriate partial computable function to start this reduction, consider what we need to show in the end. We want that if $e \notin K$ (i.e. $\varphi_e(e)\uparrow$), then $W_{f(e)}$ is finite. That is, $\varphi_{f(e)}(y)\uparrow$ for all but finitely many y, or even $\varphi_{f(e)}(y)\uparrow$ for all y. To achieve this, we watch the steps of the computation of $\varphi_e(e)$. As long as $\varphi_{e,s}(e)$ has not halted, we think that we might have $\varphi_e(e)\uparrow$, so we want to make $\varphi_{f(e)}$ diverge on (some initial segment of) its inputs - say make $\varphi_{f(e)}(0)\uparrow, \ldots, \varphi_{f(e)}(s)\uparrow$. But, if we ever see $\varphi_{e,s}(e)\downarrow$, then we want to switch to making $\varphi_{f(e)}$ converge on all of the inputs starting at s.

In the first step of the proof, we put this intuition into a partial computable function. Because the variable y in the intuitive idea above corresponds to steps in a computation, I will replace y by the more suggestive variable s.

$$\psi(x,s) = \begin{cases} 0 & \text{if } \varphi_{x,s}(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Note that in this definition, we can computably determine which case we are in. So the function ψ is partial computable, and we can fix an index e such that $\psi = \varphi_e$. That is

$$\varphi_e(x,s) = \begin{cases} 0 & \text{if } \varphi_{x,s}(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

The second step is to use the s-m-n theorem to push one of the variables down into the index. By the s-m-n theorem, we have $\varphi_e(x,s) = \varphi_{s_1^1(e,x)}(s)$, where s_1^1 is a 1-1 computable function (and, in particular, is total). Let $f(x) = s_1^1(e,x)$. Then f is a 1-1 computable function (and again, in particular, is total) and

$$\varphi_{f(x)}(s) = \begin{cases} 0 & \text{if } \varphi_{x,s}(x)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

The third step is to show that this function f gives the required reduction. In this case, we need to show that $x \in \overline{K} \Leftrightarrow f(x) \in \text{Fin}$. Again, we prove the two implications separately.

$$x \in \overline{K} \Rightarrow \varphi_x(x)\uparrow \Rightarrow \forall s(\varphi_{x,s}(x)\uparrow) \Rightarrow \forall s(\varphi_{f(x)}(s)\uparrow) \Rightarrow \text{domain}(\varphi_{f(x)}) = \emptyset \Rightarrow f(x) \in \text{Fin}$$

On the other hand

$$x \notin \overline{K} \Rightarrow \varphi_x(x)\downarrow \Rightarrow \exists s(\varphi_{x,s}(x)\downarrow)$$

At this point, we need an additional observation. If $\varphi_{x,s}(x)\downarrow$, then for all $t \geq s$, we have $\varphi_{x,t}(x)\downarrow$. So, we can continue our implications

$$x \notin \overline{K} \Rightarrow \exists s \forall t \geq s(\varphi_{x,t}(x)\downarrow) \Rightarrow \exists s \forall t \geq s(\varphi_{f(x)}(t)\downarrow) \Rightarrow [s,\infty) \subseteq \text{domain}(\varphi_{f(x)}) \Rightarrow f(x) \notin \text{Fin}$$