Computability Theory: Problem Set 1, Due Friday February 7

Problem 1. Prove that $\leq_m$ is reflexive ($A \leq_m A$ for all A) and transitive (if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$).

Problem 2. We defined the *join of sets* A and B to be

$$A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}$$

Prove that $A \leq_m A \oplus B$ and $B \leq_M A \oplus B$. (This is essentially trivial.) Finally, prove that if $A \leq_m C$ and $B \leq_m C$, then $A \oplus B \leq_m C$.

Problem 3. Let $K_1 = \{x \mid W_x \neq \emptyset\} = \{x \mid \exists y\, (\varphi_x(y)\downarrow)\}$. Prove that $K_1 \equiv_m K$.

For the next problem, we need a definition. We say that disjoint sets A and B are *computably inseparable* if there is no computable set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

Problem 4. Let $A = \{e \mid \varphi_e(e) = 0\}$ and $B = \{e \mid \varphi_e(e) = 1\}$.

4(a). Prove that A and B are computably inseparable.

4(b). Use the fact that A and B are computably inseparable to give a proof that $\mathrm{Ext} \neq \mathbb{N}$ that is different from the proof in class.

4(c). Prove that $A \equiv_m K$. (Essentially the same proof also shows that $B \equiv_m K$.)

Problem 5. Prove that Fin is not c.e. by showing that $\overline{K} \leq_m \mathrm{Fin}$. Similarly, show that Tot and Cof are not c.e.

Problem 6. Prove that every infinite c.e. set contains an infinite computable set.