

Math 5026 Homework 5 (Philosophy): Due Wednesday November 28

Problem 1. The Replacement Scheme says that for each formula $\varphi(x, y, \bar{v})$, we have an axiom of the form

$$\forall \bar{v} \forall A (\forall x \in A \exists! y \varphi(x, y, \bar{v}) \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi(x, y, \bar{v}))$$

Use the Reflection Theorem to show that we can remove the “uniqueness assumption” about the y 's. That is, ZF satisfies the stronger collection of axioms stating that

$$\forall \bar{v} \forall A (\forall x \in A \exists y \varphi(x, y, \bar{v}) \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi(x, y, \bar{v}))$$

Hint. Your set Y will be of the form $R(\beta)$ for an ordinal β . Start by fixing \bar{v} and A , and let α be greater than the ranks of these elements. Assume that $\forall x \in A \exists y \varphi(x, y, \bar{v})$. Find an appropriate β for which you can show $\forall x \in A \exists y \in R(\beta) \varphi(x, y, \bar{v})$.

For the remaining problems, let M be a countable transitive model of ZFC (or really, some suitable finite fragment of ZFC), $P = \langle P, \leq, 1_P \rangle$ be a forcing poset in M , and G be a P -generic filter over M .

Problem 2. Let $E \subseteq P$ with $E \in M$. Prove that either $G \cap E \neq \emptyset$ or $\exists q \in G \forall r \in E (q \perp r)$.

Hint. Consider the set $D = \{p \in P \mid \exists r \in E (p \leq r)\} \cup \{q \in P \mid \forall r \in E (q \perp r)\}$.

Problem 3. For $p \in P$ and $E \subseteq P$, we say E is *dense below* p if $\forall q \leq p \exists r \leq q (r \in E)$. Prove that if $E \in M$ is dense below p and $p \in G$, then $G \cap E \neq \emptyset$.

Hint. Use Problem 2.

Problem 4. For $\sigma, \tau \in M^P$, prove that $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

Problem 5. For $\tau \in M^P$, let

$$\pi = \{\langle \rho, p \rangle \mid \exists \langle \sigma, q \rangle \in \tau \exists r \in P (\langle \rho, r \rangle \in \sigma \wedge p \leq r \wedge p \leq q)\}$$

Show that $\pi_G = \bigcup \tau_G$.