Math 5026 Homework 5 (Math): Due Wednesday November 28

Problem 1. The Replacement Scheme says that for each formula $\varphi(x, y, \overline{v})$, we have an axiom of the form

$$\forall\overline{v}\,\forall A\,\big(\forall x \in A\,\exists! y\,\varphi(x, y, \overline{v}) \to \exists Y\,\forall x \in A\,\exists y \in Y\,\varphi(x, y, \overline{v})\big)$$

Use the Reflection Theorem to show that we can remove the "uniqueness assumption" about the y's. That is, ZF satisfies the stronger collection of axioms stating that

$$\forall\overline{v}\,\forall A\,\big(\forall x \in A\,\exists y\,\varphi(x, y, \overline{v}) \to \exists Y\,\forall x \in A\,\exists y \in Y\,\varphi(x, y, \overline{v})\big)$$

For the remaining problems, let M be a countable transitive model of ZFC (or really, some suitable finite fragment of ZFC), $P = \langle P, \leq, 1_P\rangle$ be a forcing poset in M, and G be a P-generic filter over M.

Problem 2. Let $E \subseteq P$ with $E \in M$. Prove that either $G \cap E \neq \emptyset$ or $\exists q \in G\,\forall r \in E\,(q \perp r)$.

Hint. Consider the set $D = \{p \in P \mid \exists r \in E\,(p \leq r)\} \cup \{q \in P \mid \forall r \in E\,(q \perp r)\}$.

Problem 3. For $p \in P$ and $E \subseteq P$, we say E is *dense below* p if $\forall q \leq p\,\exists r \leq q\,(r \in E)$. Prove that if $E \in M$ is dense below p and $p \in G$, then $G \cap E \neq \emptyset$.

Hint. Use Problem 2.

Problem 4. For $\sigma, \tau \in M^P$, prove that $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

Problem 5. For $\tau \in M^P$, let

$$\pi = \{\langle \rho, p\rangle \mid \exists\langle \sigma, q\rangle \in \tau\,\exists r \in P\,(\,\langle \rho, r\rangle \in \sigma \,\wedge\, p \leq r \,\wedge\, p \leq q)\}$$

Show that $\pi_G = \bigcup \tau_G$.