Problem 1. Which axioms of ZFC are true in $\Omega\mathbb{N}$?

Problem 2. Let $\mathbb{M}$ be a transitive class that satisfies the Comprehension Scheme and has the property that $\forall X \subseteq \mathbb{M} \exists y \in \mathbb{M} (x \subseteq y)$. Show that ZF proves that $\mathbb{M} \models \text{ZF}$.

The next problem asks you to prove the Tarski-Vaught criterion for elementary substructures. To make sure the notation is clear, $A \subseteq B$ denotes that $A$ is a substructure of $B$. That is, the domain of $A$ is contained in the domain of $B$ and for all atomic formulas $\psi(\overline{x})$ and $\overline{a} \in A$, $A \models \psi(\overline{a})$ if and only if $B \models \psi(\overline{a})$. Here, I am being lazy and writing $\overline{a} \in A$ to indicate that each element in the tuple $\overline{a}$ is an element of the domain of $A$. I will continue to use this abbreviation.

On the other hand, $A \preceq B$ denotes that $A$ is an elementary substructure of $B$. That is, $A \subseteq B$ and in addition, for all formulas $\psi(\overline{x})$ and $\overline{a} \in A$, $A \models \psi(\overline{a})$ if and only if $B \models \psi(\overline{a})$.

Problem 3. Let $L$ be a first order language and let $A, B$ be $L$-structures. Prove that if $A \subseteq B$ and for every $L$-formula $\phi(x, y)$ and $a \in A$, there is a $c \in B$ such that $B \models \phi(c, a)$ if and only if $B \models \phi(c, a)$, then $A \preceq B$. (The key point in this criterion is that you only have to look at satisfaction in the structure $B$.)

Hint. You can assume that formulas are written using only the connectives $\neg$, $\land$ and $\exists$. You need to show that for every formula $\psi(\overline{x})$ and tuple $\overline{a} \in A$, $A \models \psi(\overline{a})$ if and only if $B \models \psi(\overline{a})$. Proceed by induction on $\psi$.

The last problem is a version of the Downward Lowenheim-Skolem theorem that we will use later. You can (and should) use the Axiom of Choice when proving it.

Problem 4. Let $L$ be a countable language and let $B$ be infinite $L$ structure. Prove that for any $X \subseteq B$, there is an elementary substructure $A \subseteq B$ such that $X \subseteq A$ and $|A| \leq \max\{|X|, \omega\}$. Furthermore, if $X$ is infinite, then $|A| = |X|$.

Hint. Let $B$ be the domain of $B$. For each $L$-formula $\phi(x, \overline{y})$, define a function $f_{\phi} : B^k \to B$ (where $k =$ the length of the tuple $\overline{y}$) such that for all $\overline{b} \in B^k$, if $B \models \exists x \psi(x, \overline{b})$, then $B \models \psi(f(\overline{b}), \overline{b})$. Now use the Tarski-Vaught criterion and a bit of counting to get the desired result.