Math 5026: Problem Set 4, Due Wednesday October 31

Problem 1. Which axioms of ZFC are true in $\mathbb{ON}$?

Problem 2. Let $\mathbb{M}$ be a transitive class that satisfies the Comprehension Scheme and has the property that $\forall X \subseteq \mathbb{M}\, \exists y \in \mathbb{M}\, (x \subseteq y)$. Show that ZF proves that $\mathbb{M} \models \text{ZF}$.

The next problem asks you to prove the Tarski-Vaught criterion for elementary substructures. To make sure the notation is clear, $\mathcal{A} \subseteq \mathcal{B}$ denotes that $\mathcal{A}$ is a substructure of $\mathcal{B}$. That is, the domain of $\mathcal{A}$ is contained in the domain of $\mathcal{B}$ and for all atomic formulas $\psi(\overline{x})$ and $\overline{a} \in \mathcal{A}$, $\mathcal{A} \models \psi(\overline{a})$ if and only if $\mathcal{B} \models \psi(\overline{a})$. Here, I am being lazy and writing $\overline{a} \in \mathcal{A}$ to indicate that each element in the tuple $\overline{a}$ is an element of the domain of $\mathcal{A}$. I will continue to use this abbreviation.

On the other hand, $\mathcal{A} \preceq \mathcal{B}$ denotes that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. That is, $\mathcal{A} \subseteq \mathcal{B}$ and in addition, for all formulas $\psi(\overline{x})$ and $\overline{a} \in \mathcal{A}$, $\mathcal{A} \models \psi(\overline{a})$ if and only if $\mathcal{B} \models \psi(\overline{a})$.

Problem 3. Let $\mathcal{L}$ be a first order language and let $\mathcal{A}$, $\mathcal{B}$ be $\mathcal{L}$-structures. Prove that if $\mathcal{A} \subseteq \mathcal{B}$ and for every $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and $\overline{a} \in \mathcal{A}$,

$$\text{there is a } c \in \mathcal{B} \text{ such that } \mathcal{B} \models \varphi(c, \overline{a}) \Leftrightarrow \text{there is a } c \in \mathcal{A} \text{ such that } \mathcal{B} \models \varphi(c, \overline{a})$$

then $\mathcal{A} \preceq \mathcal{B}$. (The key point in this criterion is that you only have to look at satisfaction in the structure $\mathcal{B}$.)

Hint. You can assume that formulas are written using only the connectives $\neg$, $\wedge$ and $\exists$. You need to show that for every formula $\psi(\overline{x})$ and tuple $\overline{a} \in \mathcal{A}$, $\mathcal{A} \models \psi(\overline{a})$ if and only if $\mathcal{B} \models \psi(\overline{a})$. Proceed by induction on ψ.

The last problem is a version of the Downward Lowenheim-Skolem theorem that we will use later. You can (and should) use the Axiom of Choice when proving it.

Problem 4. Let $\mathcal{L}$ be a countable language and let $\mathcal{B}$ be infinite $\mathcal{L}$ structure. Prove that for any $X \subseteq \mathcal{B}$, there is an elementary substructure $\mathcal{A} \subseteq \mathcal{B}$ such that $X \subseteq \mathcal{A}$ and $|\mathcal{A}| \leq \max\{|X|, \omega\}$. Furthermore, if X is infinite, then $|\mathcal{A}| = |X|$.

Hint. Let B be the domain of $\mathcal{B}$. For each $\mathcal{L}$-formula $\varphi(x, \overline{y})$, define a function $f_\psi : B^k \to B$ (where $k =$ the length of the tuple $\overline{y}$) such that for all $\overline{b} \in B^k$, if $\mathcal{B} \models \exists x\, \psi(x, \overline{b})$, then $\mathcal{B} \models \psi(f(\overline{b}), \overline{b})$. Now use the Tarski-Vaught criterion and a bit of counting to get the desired result.