**Math 5026: Problem Set 3 (Philosophy), Due Monday October 22**

**Problem 1.** Let $A$ be a nonempty set of cardinals. Prove that $\sup A$ is a cardinal.

**Problem 2.** Let $\kappa$ be an infinite regular cardinal and let $\mathcal{F}$ be a family of sets with $|\mathcal{F}| < \kappa$ and $|X| < \kappa$ for each $X \in \mathcal{F}$. Prove that $|\bigcup \mathcal{F}| < \kappa$.

**Problem 3.** Let $\kappa$ be an infinite cardinal. Prove the following properties of $H(\kappa)$.

1. $H(\kappa)$ is transitive.
2. For a transitive set $x$, $x \in H(\kappa)$ if and only if $|x| < \kappa$.
3. $H(\kappa) \cap \mathbb{ON} = \kappa$
4. If $x \in H(\kappa)$, then $\bigcup x \in H(\kappa)$.
5. If $x, y \in H(\kappa)$, then $\{x, y\} \in H(\kappa)$ and $\langle x, y \rangle \in H(\kappa)$.
6. If $x \in H(\kappa)$ and $y \subseteq x$, then $y \in H(\kappa)$.

**Problem 4.** Let $\kappa$ be an infinite regular cardinal.

**4(a).** Prove that if $y \subseteq H(\kappa)$ and $|y| < \kappa$, then $y \in H(\kappa)$.

**4(b).** Prove that if $A \in H(\kappa)$, then $A \times A \in H(\kappa)$.

**4(c).** Prove that if $z \in H(\kappa)$ and $f : z \to H(\kappa)$, then $f \in H(\kappa)$ and $\mathrm{range}(f) \in H(\kappa)$.

**Hint for Problem 1.** I would break this problem into two cases. First, suppose that A has a greatest element. That is, there is a $\kappa \in A$ such that for all $\lambda \in A$, $\lambda \leq \kappa$. This case is straightforward.

Second, suppose that $A$ doesn't have a greatest element. You know that $\sup A = \alpha$ for some $\alpha \in \mathbb{ON}$. For a contradiction, suppose that $\alpha$ is not a cardinal. What does that tell you about the relationship between $|\alpha|$ and $\alpha$? How is that helpful?

**Hint for Problem 2.** Let $\lambda = |\mathcal{F}|$. Fixing a bijection between $\lambda$ and $\mathcal{F}$, you can think of $\mathcal{F} = \{X_\alpha \mid \alpha < \lambda\}$. Suppose for a contradiction that $|\bigcup \mathcal{F}| \geq \kappa$. This means you can fix an onto function $f : \bigcup \mathcal{F} \to \kappa$. Because $\lambda < \kappa$ and $\kappa$ is regular, you will arrive at the desired contradiction if you can produce a cofinal function $g : \lambda \to \kappa$. Think about how you can define $g(\alpha)$ using $f$ and $X_\alpha$ so that $g$ is the desired cofinal function.

**Hint for Problem 3.** The proof for each of these properties is very short. It might be helpful for you to start by noting a couple of general facts about transitive closures. For example, you might start by proving that (1.) if $y \in x$ or $y \subseteq x$, then $\mathrm{trcl}(y) \subseteq \mathrm{trcl}(x)$, and (2.) if $x$ is transitive, then $\mathrm{trcl}(x) = x$. These properties should help you a lot.

**Hint for Problem 4.** For 4(a), use Problem 2 to help you. For 4(b), use Problem 4(a) and Problem 3 to help you. For 4(c), use 4(a) and 4(b) to help you.