Math 5026: Problem Set 3 (Math), Due Monday October 22

Problem 1. Let A be a nonempty set of cardinals. Prove that $\sup A$ is a cardinal.

Problem 2. Let κ be an infinite regular cardinal and let $\mathcal{F}$ be a family of sets with $|\mathcal{F}| < \kappa$ and $|X| < \kappa$ for each $X \in \mathcal{F}$. Prove that $|\bigcup \mathcal{F}| < \kappa$.

Problem 3. Let κ be an infinite cardinal. Prove the following properties of $H(\kappa)$.

1. $H(\kappa)$ is transitive.
2. For a transitive set x, $x \in H(\kappa)$ if and only if $|x| < \kappa$.
3. $H(\kappa) \cap \mathbb{ON} = \kappa$
4. If $x \in H(\kappa)$, then $\bigcup x \in H(\kappa)$.
5. If $x, y \in H(\kappa)$, then $\{x, y\} \in H(\kappa)$ and $\langle x, y \rangle \in H(\kappa)$.
6. If $x \in H(\kappa)$ and $y \subseteq x$, then $y \in H(\kappa)$.

Problem 4. Let κ be an infinite regular cardinal.

4(a). Prove that if $y \subseteq H(\kappa)$ and $|y| < \kappa$, then $y \in H(\kappa)$.

4(b). Prove that if $A \in H(\kappa)$, then $A \times A \in H(\kappa)$.

4(c). Prove that if $z \in H(\kappa)$ and $f : z \to H(\kappa)$, then $f \in H(\kappa)$ and $\text{range}(f) \in H(\kappa)$.