Math 5026: Problem Set 2 (Philosophy), Due Wednesday October 3

For the first two problems, you will prove properties of the supremum notion we use for ordinals. Recall that if A is a non-empty set of ordinals, then $\sup(A) = \bigcup_{\alpha \in A} \alpha$. That is, $\beta \in \sup(A)$ if and only if there is an $\alpha \in A$ such that $\beta \in \alpha$. The reason for the notation "sup" is that $\sup(A)$ is the supremum, or least upper bound, of the ordinals in A. That is, it is the least ordinal δ such that $\alpha \leq \delta$ for all $\alpha \in A$. In the next problem, you will prove this property. To prove this fact, we give a formal definition for the least upper bound of a non-empty set of ordinals A. Let A be an arbitrary non-empty set of ordinals.

- We say an ordinal δ is an *upper bound* of A if for every $\alpha \in A$, $\alpha \leq \delta$.
- We say an ordinal δ is the *least upper bound* of A if δ is an upper bound of A and $\delta \leq \widehat{\delta}$ for every other upper bound $\widehat{\delta}$ of A.

Problem 1. Let A be a non-empty set of ordinals. Prove that $\sup(A)$ is the least upper bound of A. That is, let $\delta = \sup(A)$. In class, we proved that δ is an ordinal, so you need to prove the following two facts.

1(a). Prove that δ is an upper bound of A.

1(b). Prove that if $\widehat{\delta}$ is an upper bound of A, then $\delta \leq \widehat{\delta}$.

Problem 2. Let A and B be non-empty sets of ordinals such that for every $\alpha \in A$, there is a $\beta \in B$ such that $\alpha \leq \beta$. Prove that $\sup(A) \leq \sup(B)$.

When proving properties of ordinal arithmetic, it is sometimes useful to work directly with ordinals as linear orders and it is sometimes useful to use the recursive definitions of the arithmetic operations. For Problems 3 and 4, I think the easiest approach is to work with ordinals as linear orders. In Problem 3, you will first prove a general method for showing that $\alpha \leq \beta$ and then use this method to verify a property of ordinal addition. To set-up the general method, we need two definitions. Let α and β be ordinals.

- A function $f : \alpha \to \beta$ is *order preserving* if for all $x, y \in \alpha$, $x \leq y$ implies $f(x) \leq f(y)$.
- A function $f : \alpha \to \beta$ is *bounded* if there is an element $b \in \beta$ such that $f(a) < b$ for all $a \in \alpha$.

Problem 3. Let α and β be ordinals.

3(a). Prove that there is no one-to-one order preserving bounded function $f : \alpha \to \alpha$.

3(b). Prove that if there is a one-to-one order preserving function $f : \alpha \to \beta$, then $\alpha \leq \beta$.

3(c). Let α, β, γ and δ be ordinals such that $\alpha \leq \beta$ and $\gamma \leq \delta$. Prove that $\alpha + \gamma \leq \beta + \delta$.

For Problem 4, I would continue to work with ordinals as linear orders. In 4(a), you will prove that left cancellation holds for ordinal addition. To give some motivation for why you need to check properties like this, notice that right cancellation does not hold for ordinal addition. That is, $1+\omega = 2+\omega$ (as both are equal to ω) but you cannot cancel (i.e. subtract) the ω on each side to get $1 = 2$.

Problem 4. Let α, β, γ and δ be ordinals.

4(a). Prove that if $\alpha + \gamma = \alpha + \delta$, then $\gamma = \delta$.

4(b). Prove that if $\alpha \leq \beta$, then there is a unique δ such that $\alpha + \delta = \beta$.

The next problem concerns transfinite induction. Our version of transfinite induction said that every non-empty class of ordinals has a least element. However, that is not how we usually think of induction. In Problem 5, you will show that the form of transfinite induction from class is equivalent to statements that looks more like our standard view of induction.

Problem 5. Prove the following are equivalent.

(1) Every non-empty class $\mathbb{C} \subseteq \mathbb{ON}$ has an least element.

(2) If a class $\mathbb{A} \subseteq \mathbb{ON}$ satisfies

$$\forall \alpha \in \mathbb{ON} \left(\forall \beta < \alpha \, (\beta \in \mathbb{A}) \rightarrow \alpha \in \mathbb{A} \right)$$

then $\mathbb{A} = \mathbb{ON}$.

(3) If a class $\mathbb{A} \subseteq \mathbb{ON}$ satisfies

- $0 \in \mathbb{A}$,
- $\forall \alpha \in \mathbb{ON} \left(\alpha \in \mathbb{A} \rightarrow s(\alpha) \in \mathbb{A} \right)$, and
- $\forall \alpha \in \mathbb{ON} \left((\alpha \text{ is a limit } \wedge \forall \beta < \alpha \, (\beta \in \mathbb{A})) \rightarrow \alpha \in \mathbb{A} \right)$

then $\mathbb{A} = \mathbb{ON}$.

Problem 6 is a good example of properties that I think is easiest to prove using the recursive definition of the ordinal operations. Recall that $\alpha + \gamma$ can be defined recursively by

$$\begin{gathered} \alpha + 0 = \alpha \\ \text{If } \gamma = \delta + 1, \text{ then } \alpha + \gamma = \alpha + (\delta + 1) = (\alpha + \delta) + 1 \\ \text{If } \gamma \text{ is a limit, then } \alpha + \gamma = \sup\{\alpha + \delta \mid \delta < \gamma\} \end{gathered}$$

and that $\alpha \cdot \gamma$ can be defined recursively by

$$\begin{gathered} \alpha \cdot 0 = 0 \\ \text{If } \gamma = \delta + 1, \text{ then } \alpha \cdot \gamma = \alpha \cdot (\delta + 1) = (\alpha \cdot \delta) + \alpha \\ \text{If } \gamma \text{ is a limit, then } \alpha \cdot \gamma = \sup\{\alpha \cdot \delta \mid \delta < \gamma\} \end{gathered}$$

When you prove the properties in Problem 6 by induction on γ, break your proof into three cases corresponding to the cases in the recursive definition.

Problem 6. Fix ordinals $\alpha \leq \beta$.

6(a). Prove by induction that for all ordinals γ, $\alpha + \gamma \leq \beta + \gamma$.

6(b). Prove by induction that for all ordinals γ, $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

6(c). Prove that you cannot replace $\leq$ by $<$ in 6(b). That is, give a concrete example of ordinals $\alpha < \beta$ and $\gamma \neq 0$ such that $\alpha \cdot \gamma < \beta \cdot \gamma$ does NOT hold.

Problem 7. Define a sequence of ordinals α_n for $n \in \omega$ as follows: $\alpha_0 = \omega$ and $\alpha_{n+1} = \alpha_n$. That is, the sequence looks like

$$\omega, \quad \omega^\omega, \quad \omega^{\omega^\omega}, \quad \omega^{\omega^{\omega^\omega}}, \quad \cdots$$

Let $\varepsilon_0 = \sup(\{\alpha_n \mid n \in \omega\})$ be the limit of this sequence of ordinals. Calculate ω^{ε_0}.

Some hints for these problems

Hint for Problem 1. For each part of this problem, write down carefully what you need to show using the union definition of $\sup(A)$. The results will follow very quickly from some basic facts about ordinals and unions.

Hint for Problem 2. By Problem 1, $\sup(A)$ is the *least* upper bound of A. So, to do Problem 2, it suffices to show that $\sup(B)$ is an upper bound for A. Use the hypothesis to prove this fact.

Hint for Problem 3(a). Prove this statement by induction on α. For a contradiction, assume α is the least ordinal such that there is a one-to-one order preserving bounded function $f : \alpha \to \alpha$. Since f is bounded, fix $b \in \alpha$ such that for all $a \in A$, $f(a) < b$. To derive a contradiction (to the fact that α is the *least* ordinal for which such a function f exists), consider the function f restricted to the elements in b.

Hint for Problem 3(b). Do this problem by contradiction. Assume that $f : \alpha \to \beta$ is one-to-one and order preserving but that $\beta < \alpha$. Explain why f gives a contradiction to 3(a).

Hint for Problem 3(c). Using 3(b), it suffices to define a one-to-one order preserving function $f : \alpha + \gamma \to \beta + \delta$. Viewing $\alpha + \gamma$ as a linear order, you can think of the elements as given in the form $\langle a, 0 \rangle$ for $a \in \alpha$ and $\langle c, 1 \rangle$ for $c \in \gamma$. Similarly, viewing $\beta + \delta$ as a linear order, you can view the elements as given in the form $\langle b, 0 \rangle$ for $b \in \beta$ and $\langle d, 1 \rangle$ for $d \in \delta$. To describe the function f, you need to say where it maps $\langle a, 0 \rangle$ and $\langle c, 1 \rangle$, and check that your definition makes f one-to-one and order preserving.

Hint for Problem 4(a). It is again useful to view the elements of $\alpha + \gamma$ as having the form $\langle a, 0 \rangle$ for $a \in \alpha$ and $\langle c, 0 \rangle$ for $c \in \gamma$, and similarly for $\alpha + \delta$. Since $\alpha + \gamma = \alpha + \delta$, there is a (unique) isomorphism f between these well orders. What can you say about $f \upharpoonright \alpha$? Use your answer to analyze what f does on the γ part of $\alpha + \gamma$.

Hint for Problem 4(b). To prove that δ is unique, use 4(a). To prove that δ exists, consider the set $X = \beta \setminus \alpha$. You should think about why the set X makes sense as defined and also why $(X, \in)$ is a well order. Since $(X, \in)$ is a well order, let δ be the unique ordinal such that $(X \in) \cong \delta$. Define an isomorphism $f : \alpha + \delta \to \beta$.