Math 5026: Problem Set 2 (Math), Due Wednesday October 3

When proving properties of ordinal arithmetic, it is sometimes useful to work directly with ordinals as linear orders and it is sometimes useful to use the recursive definitions of the arithmetic operations. For Problems 1 and 2, I think the easiest approach is to work with ordinals as linear orders. In Problem 1, you will first prove a general method for showing that $\alpha \leq \beta$ and then use this method to verify a property of ordinal addition. To set-up the general method, we need two definitions. Let α and β be ordinals (or more generally, just linear orders).

- A function $f : \alpha \to \beta$ is *order preserving* if for all $x, y \in \alpha$, $x \leq y$ implies $f(x) \leq f(y)$.
- A function $f : \alpha \to \beta$ is *bounded* if there is an element $b \in \beta$ such that $f(a) < b$ for all $a \in \alpha$.

Problem 1. Let α and β be ordinals.

1(a). Prove that there is no one-to-one order preserving bounded function $f : \alpha \to \alpha$.

1(b). Prove that if there is a one-to-one order preserving function $f : \alpha \to \beta$, then $\alpha \leq \beta$.

1(c). Let α, β, γ and δ be ordinals such that $\alpha \leq \beta$ and $\gamma \leq \delta$. Prove that $\alpha + \gamma \leq \beta + \delta$.

For Problem 2, I would continue to work with ordinals as linear orders. In 2(a), you will prove that left cancellation holds for ordinal addition. To give some motivation for why you need to check properties like this, notice that right cancellation does not hold for ordinal addition. That is, $1+\omega = 2+\omega$ (as both are equal to ω) but you cannot cancel (i.e. subtract) the ω on each side to get $1 = 2$.

Problem 2. Let α, β, γ and δ be ordinals.

2(a). Prove that if $\alpha + \gamma = \alpha + \delta$, then $\gamma = \delta$.

2(b). Prove that if $\alpha \leq \beta$, then there is a unique δ such that $\alpha + \delta = \beta$.

The next problem concerns transfinite induction. Our version of transfinite induction said that every non-empty class of ordinals has a least element. However, that is not how we usually think of induction. In Problem 3, you will show that the form of transfinite induction from class is equivalent to statements that looks more like our standard view of induction.

Problem 3. Prove the following are equivalent.

(1) Every non-empty class $\mathbb{C} \subseteq \mathbb{ON}$ has an least element.

(2) If a class $\mathbb{A} \subseteq \mathbb{ON}$ satisfies

$$\forall \alpha \in \mathbb{ON} \left(\forall \beta < \alpha \, (\beta \in \mathbb{A}) \to \alpha \in \mathbb{A} \right)$$

then $\mathbb{A} = \mathbb{ON}$.

(3) If a class $\mathbb{A} \subseteq \mathbb{ON}$ satisfies

- $0 \in \mathbb{A}$,
- $\forall \alpha \in \mathbb{ON}\,\big(\alpha \in \mathbb{A} \to s(\alpha) \in \mathbb{A}\big)$, and
- $\forall \alpha \in \mathbb{ON}\,\big((\alpha \text{ is a limit } \wedge \forall \beta < \alpha\,(\beta \in \mathbb{A})) \to \alpha \in \mathbb{A}\big)$

then $\mathbb{A} = \mathbb{ON}$.

Problem 4 is a good example of properties that I think is easiest to prove using the recursive definitions of the ordinal operations. Recall that $\alpha + \gamma$ can be defined recursively by

$$\alpha + 0 = \alpha$$
$$\text{If } \gamma = \delta + 1, \text{ then } \alpha + \gamma = \alpha + (\delta + 1) = (\alpha + \delta) + 1$$
$$\text{If } \gamma \text{ is a limit, then } \alpha + \gamma = \sup\{\alpha + \delta \mid \delta < \gamma\}$$

and that $\alpha \cdot \gamma$ can be defined recursively by

$$\alpha \cdot 0 = 0$$
$$\text{If } \gamma = \delta + 1, \text{ then } \alpha \cdot \gamma = \alpha \cdot (\delta + 1) = (\alpha \cdot \delta) + \alpha$$
$$\text{If } \gamma \text{ is a limit, then } \alpha \cdot \gamma = \sup\{\alpha \cdot \delta \mid \delta < \gamma\}$$

When you prove the properties in Problem 4 by induction on γ, break your proof into three cases corresponding to the cases in the recursive definition.

Problem 4. Fix ordinals $\alpha \leq \beta$.

4(a). Prove by induction that for all ordinals γ, $\alpha + \gamma \leq \beta + \gamma$.

4(b). Prove by induction that for all ordinals γ, $\alpha \cdot \gamma \leq \beta \cdot \gamma$.

4(c). Prove that you cannot replace $\leq$ by $<$ in 4(b). That is, give a concrete example of ordinals $\alpha < \beta$ and $\gamma \neq 0$ such that $\alpha \cdot \gamma < \beta \cdot \gamma$ does NOT hold.

Problem 5. Do Exercise I.9.53 in the textbook.

Problem 6. Do Exercise I.9.54 in the textbook.