Set Theory: Problem Set 1 (Philosophy), Due Friday Sept 14

Problem 1. Use our set theoretic definition of pairs to prove that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.

Let A and B be sets and let $f : A \to B$ be a function between them. Three important properties that functions can have are:

- f is *one-to-one* if $a_0 \neq a_1$ implies that $f(a_0) \neq f(a_1)$,
- f is *onto* if for every $b \in B$, there is an $a \in A$ such that $f(a) = b$, and
- f is a *bijection* if f is both one-to-one and onto.

If we have two functions $f : A \to B$ and $g : B \to C$, the *composition function* $g \circ f : A \to B$ is defined by $(g \circ f)(a) = g(f(a))$. That is, $g \circ f$ takes an input from a, applies f to a to produce an element $f(a) \in B$, and then applies g to $f(a)$ to produce an element $g(f(a)) \in C$.

Problem 2. Let $f : A \to B$ and $g : B \to C$ be functions.
2(a). Prove that if f and g are one-to-one, then so is $g \circ f$.
2(b). Prove that if f and g are onto, then so is $g \circ f$.
2(c). Prove that if f and g are bijections, then so is $g \circ f$.

For the next problem, recall that set theoretically, $2 = \{0, 1\}$. Fix an arbitrary set A. For $Y \subseteq A$, the *characteristic function of* Y is denoted $\chi_Y : A \to 2$ and is defined by

$$\chi_Y(a) = \begin{cases} 0 & \text{if } a \notin Y \\ 1 & \text{if } a \in Y \end{cases}$$

Problem 3. Let A be a set. Prove that the function $\Delta : \mathcal{P}(A) \to 2^A$ given by $\Delta(Y) = \chi_Y$ is a bijection.

Problem 4. Let A be a set and let $g : A \to \mathcal{P}(A)$. Prove that g cannot be onto by showing that $Y = \{a \in A \mid a \notin g(a)\}$ is not in the range of g.

Hint. For a contradiction, assume $g(b) = Y$ for some $b \in A$. Is $b \in Y$?

Problem 5. Let $(A, <_A)$ and $(B, <_B)$ be well orders. Define $<_{A\times B}$ on $A \times B$ by

$$\langle a_0, b_0\rangle <_{A\times B} \langle a_1, b_1\rangle \Leftrightarrow (b_0 <_B b_1) \vee (b_0 = b_1 \wedge a_0 <_A a_1)$$

(The linear order $<_{A\times B}$ is a called the *reverse lexicographic order*.) Prove that $(A \times B, <_{A\times B})$ is a well order. You can assume it is a linear order and just show it is well founded.

The last problem is the heart of what is called the Schroeder-Bernstein Theorem. For this problem, the following notation is useful. Let $f : X \to Y$ be a function. For $Z \subseteq X$, let

$$f[Z] = \{y \in Y \mid \exists z \in Z\,(f(z) = y)\}$$

That is, $f[Z] = \mathrm{range}(f \restriction Z)$.

Problem 6. Let A and B be sets such that there are one-to-one functions $f : A \to B$ and $g : B \to A$. Prove that there is a bijection $h : A \to B$.

Hint. Fix one-to-one functions $f : A \to B$ and $g : B \to A$. We need to define a bijection $h : A \to B$. Define decreasing sequences of subsets of A and B indexed by $\mathbb{N}$

$$\begin{aligned} A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \\ B = B_0 \supseteq B_1 \supseteq B_2 \supseteq A_3 \supseteq \cdots \end{aligned}$$

by $A_0 = A$, $B_0 = B$, $A_{n+1} = g[B_n]$ and $B_{n+1} = f[A_n]$.

Step 1. Consider A_0, A_1, A_2, B_0, B_1 and B_2.

(a) Prove that f gives a bijection between $A_0 \setminus A_1$ and $B_1 \setminus B_2$.

(b) Analogously, prove that g gives a bijection between $B_0 \setminus B_1$ and $A_1 \setminus A_2$.

Step 2. Using Step 1, show that $h : A_0 \setminus A_2 \to B_0 \setminus B_2$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_0 \setminus A_1 \\ g^{-1}(x) & \text{if } x \in A_1 \setminus A_2 \end{cases}$$

is a bijection.

Step 3. Using essentially the same arguments, show that for any $n \in \mathbb{N}$:

(a) f gives a bijection between $A_{2n} \setminus A_{2n+1}$ and $B_{2n+1} \setminus B_{2n+2}$ and

(b) g gives a bijection between $B_{2n} \setminus B_{2n+1}$ and $A_{2n+1} \setminus A_{2n+2}$.

Step 4. Let $A_\infty = \bigcap_{n\in\mathbb{N}} A_n$ and $B_\infty = \bigcap_{n\in\mathbb{N}} B_n$. Prove that f gives a bijection between A_∞ and B_∞.

Step 5. Prove that $h : A \to B$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_{2n} \setminus A_{2n+1} \text{ for some } n \\ g^{-1}(x) & \text{if } x \in A_{2n+1} \setminus A_{2n+2} \text{ for some } n \\ f(x) & \text{if } x \in A_\infty \end{cases}$$

is a bijection.