Set Theory: Problem Set 1 (Math), Due Friday Sept 14

Problem 1. Use our set theoretic definition of pairs to prove that $\langle a, b\rangle = \langle c, d\rangle$ if and only if $a = c$ and $b = d$.

Problem 2. Let A be a set and let $g : A \to \mathcal{P}(A)$. Prove that g cannot be onto by showing that $Y = \{a \in A \mid a \notin g(a)\}$ is not in the range of g.

Hint. For a contradiction, assume $g(b) = Y$ for some $b \in A$. Is $b \in Y$?

Problem 3. Let $(A, <_A)$ and $(B, <_B)$ be well orders. Define $<_{A\times B}$ on $A \times B$ by

$$\langle a_0, b_0\rangle <_{A\times B} \langle a_1, b_1\rangle \Leftrightarrow (b_0 <_B b_1) \vee (b_0 = b_1 \wedge a_0 <_A a_1)$$

(The linear order $<_{A\times B}$ is a called the *reverse lexicographic order*.) Prove that $(A \times B, <_{A\times B})$ is a well order. You can assume it is a linear order and just show it is well founded.

Problem 4. Consider $2^{\mathbb{N}}$, the set of all functions from $\mathbb{N}$ to $\{0, 1\}$. If $f, g \in 2^{\mathbb{N}}$ and $f \neq g$, then there is a least n such that $f(n) \neq g(n)$. Define the following linear order on $2^{\mathbb{N}}$.

$$f \prec g \Leftrightarrow f(n) <_{\mathbb{N}} g(n) \text{ where } n \text{ is least such that } f(n) \neq g(n)$$

Prove that $\prec$ is not a well order on $2^{\mathbb{N}}$.

Hint. Consider the functions $g_k(x)$ for $k \in \mathbb{N}$ given by

$$g_k(x) = \begin{cases} 0 & \text{if } x < k \\ 1 & \text{if } x \geq k \end{cases}$$

For the next problem, we need a bit of terminology. Let $(A, \leq_A)$ be a well order such that $|A| \geq 2$ and let 0_A denote the $\leq_A$-least element of A. Let $(B, \leq_B)$ be a nonempty well order. For a function $f : B \to A$, the *support of* f is

$$\text{support}(f) = \{b \in B \mid f(b) \neq 0_A\}$$

We are interested in the functions $f : B \to A$ with finite support.

$$\mathcal{F}(B, A) = \{f \in A^B \mid \text{support}(f) \text{ is finite}\}$$

Define a linear order $<_{\mathcal{F}}$ on $\mathcal{F}(B, A)$ as follows. For $f, g \in \mathcal{F}(B, A)$

$$f <_{\mathcal{F}} g \Leftrightarrow f \neq g \text{ and } f(b) <_A g(b) \text{ where } b = \max\{x \in B \mid f(x) \neq g(x)\}$$

If $f \neq g$, then because the supports of f and g are finite, the set $\{x \in B \mid f(x) \neq g(x)\}$ is finite and nonempty. Therefore, it has a $<_B$-maximum element, so the condition above is well defined.

Problem 5. Prove that $<_{\mathcal{F}}$ is a well order on $\mathcal{F}(B, A)$. You can assume it is a linear order and just show it is well founded.

The last problem is the heart of what is called the Schroeder-Bernstein Theorem. For this problem, the following notation is useful. Let $f : X \to Y$ be a function. For $Z \subseteq X$, let

$$f[Z] = \{y \in Y \mid \exists z \in Z\,(f(z) = y)\}$$

That is, $f[Z] = \text{range}(f \upharpoonright Z)$.

Problem 6. Let A and B be sets such that there are one-to-one functions $f : A \to B$ and $g : B \to A$. Prove that there is a bijection $h : A \to B$.

Hint. Fix one-to-one functions $f : A \to B$ and $g : B \to A$. We need to define a bijection $h : A \to B$. Define decreasing sequences of subsets of A and B indexed by $\mathbb{N}$

$$\begin{aligned} A = A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots \\ B = B_0 \supseteq B_1 \supseteq B_2 \supseteq A_3 \supseteq \cdots \end{aligned}$$

by $A_0 = A$, $B_0 = B$, $A_{n+1} = g[B_n]$ and $B_{n+1} = f[A_n]$.

Step 1. Consider A_0, A_1, A_2, B_0, B_1 and B_2.

(a) Prove that f gives a bijection between $A_0 \setminus A_1$ and $B_1 \setminus B_2$.

(b) Analogously, prove that g gives a bijection between $B_0 \setminus B_1$ and $A_1 \setminus A_2$.

Step 2. Using Step 1, show that $h : A_0 \setminus A_2 \to B_0 \setminus B_2$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_0 \setminus A_1 \\ g^{-1}(x) & \text{if } x \in A_1 \setminus A_2 \end{cases}$$

is a bijection.

Step 3. Using essentially the same arguments, show that for any $n \in \mathbb{N}$:

(a) f gives a bijection between $A_{2n} \setminus A_{2n+1}$ and $B_{2n+1} \setminus B_{2n+2}$ and

(b) g gives a bijection between $B_{2n} \setminus B_{2n+1}$ and $A_{2n+1} \setminus A_{2n+2}$.

Step 4. Let $A_\infty = \bigcap_{n\in\mathbb{N}} A_n$ and $B_\infty = \bigcap_{n\in\mathbb{N}} B_n$. Prove that f gives a bijection between A_∞ and B_∞.

Step 5. Prove that $h : A \to B$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_{2n} \setminus A_{2n+1} \text{ for some } n \\ g^{-1}(x) & \text{if } x \in A_{2n+1} \setminus A_{2n+2} \text{ for some } n \\ f(x) & \text{if } x \in A_\infty \end{cases}$$

is a bijection.