

Math 5026 Homework 2: Due Wednesday October 8

These problems are intended to give you an introduction to some special ordinals called epsilon numbers. Throughout this assignment, all the exponentiation is ordinal exponentiation. I have broken down the analysis of these numbers into small steps, so the proofs of some (but not all) of the problems below are short.

An ordinal α is called an *epsilon number* if $\alpha = \omega^\alpha$. The goal is to show that epsilon numbers exist and in fact, there are lots (i.e. class many) of them. The smallest epsilon number is called ε_0 and is given by

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$$

Problems 1. Prove that ε_0 is an epsilon number (i.e. $\varepsilon_0 = \omega^{\varepsilon_0}$).

Problems 2. Prove that every epsilon number is a limit ordinal.

Problems 3. Prove that if $\alpha < \beta$, then $\omega^\alpha < \omega^\beta$.

Problems 4. Let $\alpha = \beta + 1$ be a successor ordinal. Prove that

$$\alpha < \omega^\alpha < \omega^{\omega^\alpha} < \dots$$

Problems 5. Let α be a successor ordinal. Prove that

$$\beta = \sup\{\alpha, \omega^\alpha, \omega^{\omega^\alpha}, \dots\}$$

is an epsilon number and that it is the least epsilon number greater than α .

To make the next problem easier to state, define a function $\mathbb{F} : \mathbb{ON} \rightarrow \mathbb{ON}$ by

$$\mathbb{F}(\alpha) = \sup\{\alpha + 1, \omega^{\alpha+1}, \omega^{\omega^{\alpha+1}}, \dots\}$$

By Problem 5, you know that $\mathbb{F}(\alpha)$ is the least epsilon number greater than $\alpha + 1$. However, notice that \mathbb{F} is not one-to-one.

Problems 6. Define the ordinals ε_α for $\alpha \in \mathbb{ON}$ by

$$\begin{aligned}\varepsilon_0 &= \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\} \text{ as above} \\ \varepsilon_{\beta+1} &= \mathbb{F}(\varepsilon_\beta) \\ \varepsilon_\alpha &= \sup\{\varepsilon_\gamma \mid \gamma < \alpha\} \text{ for limit } \alpha\end{aligned}$$

Prove that each ε_α is an epsilon number and that this sequence gives all the epsilon numbers in (strictly) increasing order. (That is, ε_α is the α -th epsilon number.)

Problem 7. Let $\alpha \geq 1$ be an ordinal. Let $\{\beta_\delta \mid \delta < \alpha\}$ be a sequence of ordinals indexed by α and let $\beta = \sup\{\beta_\delta \mid \delta < \alpha\}$. Prove that if $|\beta_\delta| \leq |\alpha|$ for all $\delta < \alpha$, then $|\beta| \leq |\alpha|$.

(For the problems that follow, it will be useful to use the following (simpler) variant of Problem 7. Let α be an infinite ordinal, let $\{\beta_\delta \mid \delta \in \omega\}$ be a set of ordinals with $\beta = \sup\{\beta_\delta \mid \delta \in \omega\}$. If $|\beta_\delta| \leq |\alpha|$ for all $\delta < \omega$, then $|\beta| \leq |\alpha|$. Once you've done Problem 7, the proof of this variant will be evident and you are welcome to use this variant of Problem 7 below without additional proof.)

Problem 8. Prove that for each infinite ordinal γ , $|\omega^\gamma| = |\gamma|$.

Problems 9. Prove that for each $n \in \omega$, ε_n is countable.

Problems 10. Prove that for $\alpha \geq \omega$, $|\varepsilon_\alpha| = |\alpha|$.

Problems 11. Let κ be an uncountable cardinal. Prove that κ is an epsilon number and that $\kappa = \varepsilon_\kappa$ (i.e. that there are κ many epsilon numbers below κ).

Hints for this homework

Problem 1. Since $\alpha \leq \omega^\alpha$ for every ordinal α , you get that $\varepsilon_0 \leq \omega^{\varepsilon_0}$ for free. To show $\omega^{\varepsilon_0} \leq \varepsilon_0$, notice that ε_0 is a limit ordinal and use the limit case of the definition of ordinal exponentiation.

Problem 2. This problem is a very short consequence of Exercise I.9.53 from last homework.

Problem 3. Fix α and prove $\omega^\alpha < \omega^\beta$ by induction on all ordinals $\beta > \alpha$. The base case is when $\beta = \alpha + 1$. For the successor step, you can assume $\beta = \gamma + 1$ (with $\gamma > \alpha$) and $\omega^\alpha < \omega^\gamma$. For the limit step, you can assume $\beta > \alpha$ is a limit ordinal and $\omega^\alpha < \omega^\gamma$ for all γ such that $\alpha < \gamma < \beta$.

Problem 4. Use Problem 2 to prove the first inequality. Then use Problem 3 to prove the remaining inequalities.

Problem 5. Showing that β is an epsilon number is essentially the same as Problem 1. To show β is the least epsilon number greater than α , assume for a contradiction that there is an epsilon number γ such that $\alpha \leq \gamma < \beta$. Use Problem 4 to explain why γ cannot be equal to any of α , ω^α , ω^{ω^α} and so on. Therefore, γ must sit in the gap between two ordinals from this list. Use the fact that γ is an epsilon number and Problem 3 to derive a contradiction.

Problem 6. Notice that you already know ε_0 and $\varepsilon_{\beta+1}$ are epsilon numbers by Problems 1 and 5. Therefore, you only need to show that ε_α is an epsilon number when α is a limit. Since ε_α is a limit ordinal, look at the limit case of the definition of ordinal exponentiation to help you show that $\varepsilon_\alpha \leq \alpha$.

To show that $\beta < \alpha$ implies $\varepsilon_\beta < \varepsilon_\alpha$, fix β and proceed by transfinite induction on $\alpha > \beta$. The successor case follows almost immediately from the definition of \mathbb{F} and Problem 5. The limit case follows almost immediately from the definition of ε_α for limit α and the induction hypothesis.

Problem 7. If α is finite, then $\alpha = n + 1$ for some $n \in \omega$ and $\beta = \beta_n$ so the result follows because $|\beta_n| \leq |\alpha|$. If α is infinite, use the fact that $|\alpha| \times |\alpha| = |\alpha|$.

Problem 8. Prove this by induction on γ . Use Problem 7 to help you. That is, for each of the cases (the base case when $\gamma = \omega$, the successor case when $\gamma = \xi + 1$ with $\omega \leq \xi$, and the limit case when γ is an infinite limit ordinal with $\omega < \gamma$), write down the inductive definition of ω^γ and you should be able to use a suitable form (or variant) of Problem 7 to help you.

Problem 9. Proceed by induction on n . Use Problems 7 (or its variant) and 8 to help you.

Problem 10. Proceed by induction on $\alpha \geq \omega$. Use Problems 7 (or its variant), 8 and 9 to help you.

Problem 11. Use Exercise I.9.53 from the last homework assignment to conclude that $\kappa = \omega^\delta$ for some $\delta \leq \kappa$. Use Problem 8 to explain why $\delta = \kappa$, and hence why κ is an epsilon number. Use Problem 10 to explain why $\kappa = \varepsilon_\kappa$ (as opposed to $\kappa = \varepsilon_\alpha$ for some $\alpha < \kappa$).