CHAPTER 3

# Relations and Partitions

Given a set of objects, we may want to say that certain pairs of objects are related in some way. For example, we may say that two people are related if they have the same citizenship or the same blood type, or if they like the same kinds of food. If $a$ and $b$ are integers, we might say that $a$ is related to $b$ when $a$ divides $b$. In this chapter we will study the idea of "is related to" by making precise the notion of a relation and then concentrating on certain relations called equivalence relations. The last two sections of the chapter introduce order relations and the theory of graphs.

## 3.1 Cartesian Products and Relations

When we speak of a relation on a set, we identify the notion of "$a$ is related to $b$" with the ordered pair $(a, b)$. For the set of all people, if Phoebe and Monica were born on the same day of the year, then the pair (Phoebe, Monica) is in the relation "has the same birthday as." Thus a relation may be defined simply as a set of ordered pairs.

DEFINITIONS Let $A$ and $B$ be sets. $R$ is a **relation from $A$ to $B$** iff $R$ is a subset of $A \times B$. A relation from $A$ to $A$ is called a **relation on $A$.**

If $(a, b) \in R$, we write $\boldsymbol{a \, R \, b}$ and say $a$ is **$R$-related** (or simply **related**) to $b$. If $(a, b) \notin R$, we write $\boldsymbol{a \not R \, b}$.

Examples. If $A = \{-1, 2, 3, 4\}$ and $B = \{1, 2, 4, 5, 6\}$, let

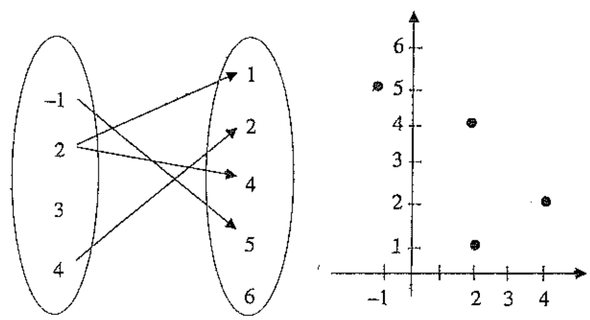
$$R = \{(-1, 5), (2, 4), (2, 1), (4, 2)\},$$
$$S = \{(5, 2), (4, 3), (1, 3)\}, \text{ and}$$
$$T = \{(-1, 3), (2, 3), (4, 4)\}.$$

Then $R$ is a relation from $A$ to $B$, $S$ is a relation from $B$ to $A$ and the set $T$ is a relation on $A$.

We could describe the relation $R$ by writing $-1\,R\,5$, $2\,R\,4$, $2\,R\,1$, and $4\,R\,2$. Since $(3, 5) \notin R$, we write $3 \not R\, 5$. We can also describe $R$ by listing the pairs of $R$ in a two-column table, by displaying the relation with an arrow diagram, or by drawing the graph of $R$ as in Figure 3.1.1.

| | |
|---|---|
| −1 | 5 |
| 2 | 4 |
| 2 | 1 |
| 4 | 2 |

(a) Table for $R$




(b) Arrow diagram for $R$ (c) Graph of $R$

**Figure 3.1.1**

An equation, inequality, expression, or graph is often used to describe a relation, especially when listing all pairs is impractical or impossible. For example, the relation $LT = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x < y\}$ is the familiar "less than" relation on $\mathbb{R}$, since $x\,LT\,y$ iff $x < y$. The graph of $LT$ is shown (shaded) in Figure 3.1.2.

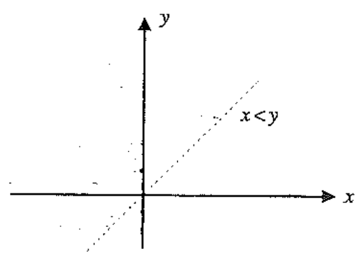
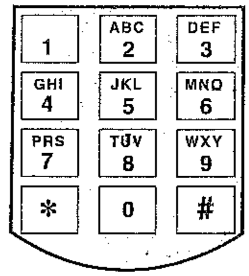



**Figure 3.1.2**

You have worked with the graphs of relations in previous courses, because, as we will see in Chapter 4, functions are relations that satisfy an additional condition.

**Example.** The phone faceplate pictured on the next page may be used to define a relation from the set of digits $\Delta = \{0, 1, 2, \ldots, 9\}$ to the set of 26 letters $\Gamma = \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots\}$. The relation $R$ defined by "appear on the same phone button" is a subset of $\Delta \times \Gamma$ containing 24 pairs. The pair $(4, \mathsf{G}) \in R$ since 4 and G appear on the same button. Likewise, $9\,R\,\mathsf{Y}$ and $6\,R\,\mathsf{M}$ are true. $(3, \mathsf{T}) \notin R$ since 3 and T do not appear together. Also $1 \not R\, \mathsf{E}$ and $4 \not R\, \mathsf{P}$ are true.




Consider the relation $S$ on the set $\mathbb{N} \times \mathbb{N}$ given by $(m, n)\, S\, (k, j)$ iff $m + n = k + j$. Then $(3, 17)\, S\, (12, 8)$, but $(5, 4)$ is not $S$-related to $(6, 15)$. Notice that $S$ is a relation from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$ and consists of ordered pairs whose entries are themselves ordered pairs. For this reason, the description above is somewhat simpler than defining $S$ with set notation:

$$S = \{((m, n), (k, j)) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) : m + n = k + j\}.$$

The empty set $\varnothing$ and the set $A \times B$ are relations from $A$ to $B$. In general, there are many different relations from a set $A$ to a set $B$ because *every* subset of $A \times B$ is a relation from $A$ to $B$. In Exercise 12 you are asked to prove that if $A$ has $m$ elements and $B$ has $n$ elements, then there are $2^{mn}$ different relations from $A$ to $B$.

DEFINITIONS The **domain** of the relation $R$ from $A$ to $B$ is the set

$$\text{Dom}(R) = \{x \in A : \text{there exists } y \in B \text{ such that } x\, R\, y\}.$$

The **range** of the relation $R$ is the set

$$\text{Rng}(R) = \{y \in B : \text{there exists } x \in A \text{ such that } x\, R\, y\}.$$

Thus the domain of $R$ is the *set of all first coordinates of ordered pairs in* $R$, and the range of $R$ is the *set of all second coordinates.* By definition, $\text{Dom}(R) \subseteq A$ and $\text{Rng}(R) \subseteq B$.

For the relation $R = \{(-1, 5), (2, 4), (2, 1), (4, 2)\}$, $\text{Dom}(R) = \{-1, 2, 4\}$ and $\text{Rng}(R) = \{1, 2, 4, 5\}$. For the relation $LT$ on $\mathbb{R}$, where $x\, LT\, y$ iff $x < y$, both the domain and range are $\mathbb{R}$. For the relation defined by "appears on same phone button," the domain is $\{2, 3, 4, 5, 6, 7, 8, 9\}$ and the range is the set of all capital letters except Q and Z.

*Every set of ordered pairs is a relation.* If $M$ is any set of ordered pairs, then $M$ is a relation from $A$ to $B$, where $A$ and $B$ are any sets for which $\text{Dom}(M) \subseteq A$ and $\text{Rng}(M) \subseteq B$.

Example. Let $S = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: \dfrac{x^2}{324} + \dfrac{y^2}{64} \leq 1\right\}$. The graph of $S$ is the shaded area in Figure 3.1.3. The domain is $[-18, 18]$ and the range is $[-8, 8]$.

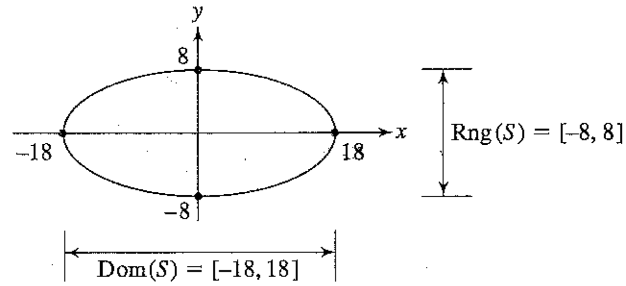



**Figure 3.1.3**

We can use a **directed graph** or **digraph** to represent a relation $R$ on a small finite set $A$. We think of the objects in $A$ as points (called **vertices**) and the relation $R$ as telling us which vertices are connected by **arcs**. Arcs are drawn as arrows: There is an arc from vertex $a$ to vertex $b$ iff $(a, b) \in R$. An arc from a vertex to itself is called a **loop**. For example, let $A = \{2, 5, 6, 12\}$ and $R = \{(6, 12), (2, 6), (2, 12), (6, 6), (12, 2)\}$. The digraph for $R$ is given in Figure 3.1.4.

The digraph of the relation "divides" on the set $\{3, 6, 9, 12\}$ has a loop at each vertex, as shown in Figure 3.1.5.

> DEFINITION For any set $A$, the relation $I_A = \{(x, x): x \in A\}$ is called the **identity relation on** $A$.

For $A = \{1, 2, a, b\}$, $I_A = \{(1, 1), (2, 2), (a, a), (b, b)\}$. Clearly, for any set $A$, Dom $(I_A) = A$ and Rng $(I_A) = A$. The graph of the identity relation on $[-2, \infty)$ is shown in Figure 3.1.6.

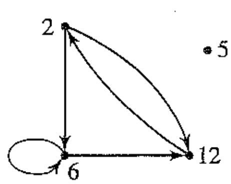



**Figure 3.1.4**

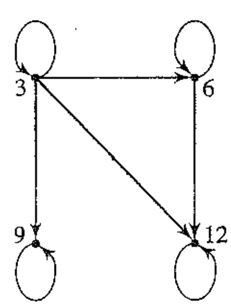



**Figure 3.1.5**

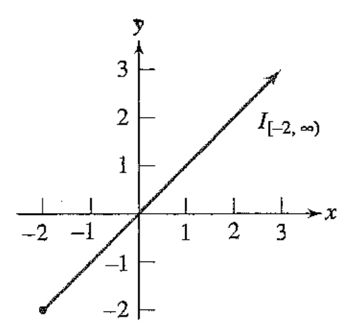
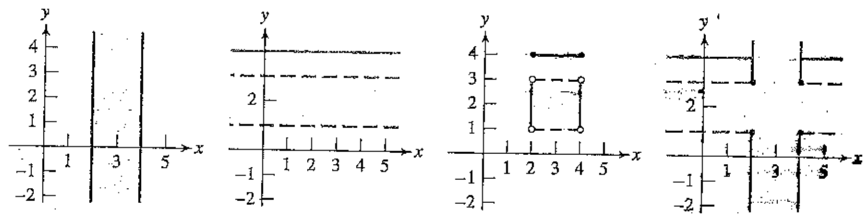



**Figure 3.1.6**

The remainder of this section is devoted to methods of constructing new relations from given relations. These ideas are important in the study of relations, and will be used again when we study functions.

Since relations from set $A$ to set $B$ are subsets of $A \times B$, the union and intersection of two relations from $A$ to $B$ are again relations from $A$ to $B$.

Example. Let $X = [2, 4]$ and $Y = (1, 3) \cup \{4\}$. Let $S$ be the relation on $\mathbb{R}$ defined by $x\,S\,y$ iff $x \in X$, and let $T$ be the relation on $\mathbb{R}$ defined by $x\,T\,y$ iff $y \in Y$. The graphs of $S$ and $T$ are given in Figures 3.1.7(a) and (b). Figure 3.1.7(c) shows the graph of $S \cap T$. Note that $S = X \times \mathbb{R}$, $T = \mathbb{R} \times Y$, and $S \cap T = X \times Y$. Figure 3.1.7(d) shows the graph of $S \cup T$.




(a) $S$ (b) $T$ (c) $S \cap T$ (d) $S \cup T$

**Figure 3.1.7**

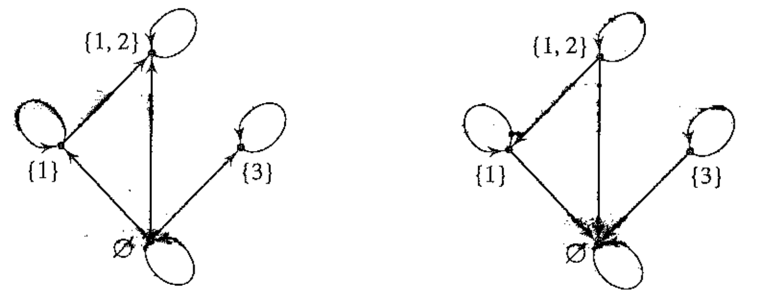
> DEFINITION If $R$ is a relation from $A$ to $B$, then the **inverse** of $R$ is the relation
>
> $$R^{-1} = \{(y, x) \colon (x, y) \in R\}.$$

Since inversion is a matter of switching the order of each pair in a relation, if $R$ is a relation from $A$ to $B$, then $R^{-1}$ is a relation from $B$ to $A$.

Examples. The inverse of the relation $R = \{(1, b), (1, c), (2, c)\}$ is the relation $R^{-1} = \{(b, 1), (c, 1), (c, 2)\}$. For any set $A$, the inverse of $I_A$ is $I_A$ itself. For the real numbers, the inverse of the "less than" relation $LT = \{(x, y) \in \mathbb{R} \times \mathbb{R} \colon x < y\}$ is the "greater than" relation on $\mathbb{R}$ because

$$\begin{aligned}(x, y) \in LT^{-1} &\text{ iff } (y, x) \in LT\\ &\text{ iff } y < x\\ &\text{ iff } x > y.\end{aligned}$$

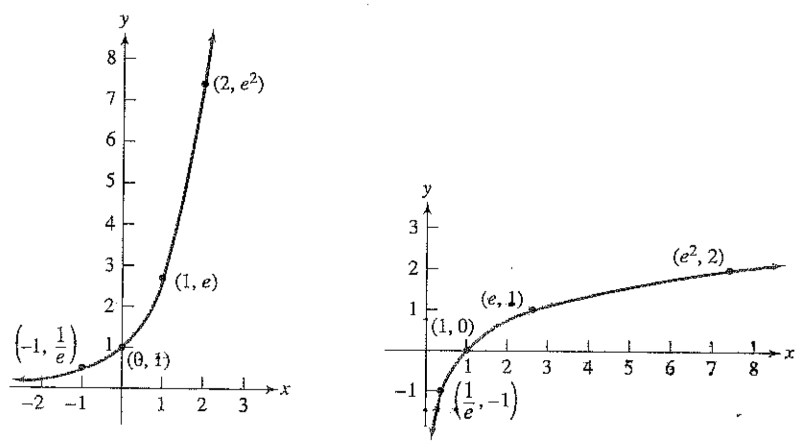
In case $R$ is a relation on $A$, the digraph of $R^{-1}$ is obtained from the digraph of $R$ by copying all the loops and arcs, but reversing the direction of the arrows for arcs. Figure 3.1.8 shows the digraphs of $R$ and $R^{-1}$, where $R$ is the relation $\subseteq$ on the set $\{\varnothing, \{1\}, \{3\}, \{1, 2\}\}$.




(a) $\subseteq$ (b) $\subseteq^{-1}$

**Figure 3.1.8**

Example. Let EXP be the relation on $\mathbb{R}$ given by $x$ EXP $y$ iff $y = e^x$. The inverse of EXP is given by $x\ \mathrm{EXP}^{-1}\ y$ iff $x = e^y$. We know that $x = e^y$ iff $y = \ln x$ iff $x \ln y$, where ln is the natural logarithm. Thus, the inverse of EXP is the relation ln. The familiar graphs of EXP and ln are given in Figure 3.1.9.




(a) $x$ EXP $y$: $y = e^x$ (b) $\ln = \mathrm{EXP}^{-1}$: $y = \ln x$

**Figure 3.1.9**

In the previous example, Dom (EXP) $= \mathbb{R}$ and Rng (EXP) $= (0, \infty)$, while Dom (ln) $= (0, \infty)$ and Rng (ln) $= \mathbb{R}$. The next theorem says that this switch of the domain and range of a relation to the range and domain of inverse relation always happens.

**Theorem 3.1.2** Let $R$ be a relation from $A$ to $B$.

**(a)** Dom $(R^{-1})$ = Rng $(R)$.
**(b)** Rng $(R^{-1})$ = Dom $(R)$.

Proof.

**(a)** $b \in \text{Dom}(R^{-1})$ iff there exists $a \in A$ such that $(b, a) \in R^{-1}$ iff there exists $a \in A$ such that $(a, b) \in R$ iff $b \in \text{Rng}(R)$.

**(b)** The proof is similar to the proof for part (a). ∎

Given a relation from $A$ to $B$ and another from $B$ to $C$, composition is a method of constructing a relation from $A$ to $C$.

> DEFINITION Let $R$ be a relation from $A$ to $B$, and let $S$ be a relation from $B$ to $C$. The **composite** of $R$ and $S$ is
>
> $$S \circ R = \{(a, c): \text{there exists } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}.$$

The relation $S \circ R$ is a relation from $A$ to $C$ since $S \circ R \subseteq A \times C$. It is always true that $\text{Dom}(S \circ R) \subseteq \text{Dom}(R)$ but it is not always true that $\text{Dom}(S \circ R) = \text{Dom}(R)$. (See Exercise 9.)

We have adopted the right-to-left notation for $S \circ R$ that is commonly used in analysis courses. To determine $S \circ R$, you need to remember that $R$ is the relation from the first set to the second and $S$ is the relation from the second set to the third. Thus, to determine $S \circ R$, we apply the relation $R$ first and then $S$.

Example. Let $A = \{1, 2, 3, 4, 5\}$, and $B = \{p, q, r, s, t\}$, and $C = \{x, y, z, w\}$. Let $R$ be the relation from $A$ to $B$:

$$R = \{(1, p), (1, q), (2, q), (3, r), (4, s)\}$$

and $S$ the relation from $B$ to $C$:

$$S = \{(p, x), (q, x), (q, y), (s, z), (t, z)\}.$$

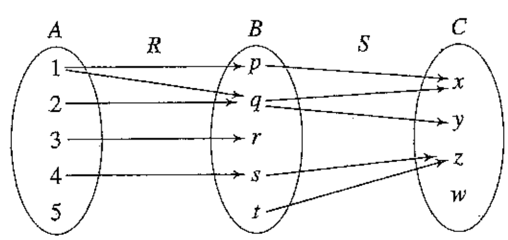



Figure 3.1.10

These relations are illustrated in Figure 3.1.10 by arrows from one set to another. An element $a$ from $A$ is related to an element $c$ from $C$ under $S \circ R$ if there is at least one "intermediate" element $b$ of $B$ such that $(a, b) \in R$ *and* $(b, c) \in S$.

For example, since $(1, p) \in R$ and $(p, x) \in S$, then $(1, x) \in S \circ R$. By following all possible paths along the arrows from $A$ to $B$ and $B$ to $C$ in Figure 3.1.10, we have

$$S \circ R = \{(1, x), (1, y), (2, x), (2, y), (4, z)\}.$$

If $R$ is a relation from $A$ to $B$, and $S$ is a relation from $B$ to $A$, then $R \circ S$ and $S \circ R$ are both defined, but you should not expect that $R \circ S = S \circ R$. Even when $R$ and $S$ are relations on the same set, it may happen that $R \circ S \neq S \circ R$.

Example. Let $R = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x + 1\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2\}$. Then

$$\begin{aligned}
R \circ S &= \{(x, y): (x, z) \in S \text{ and } (z, y) \in R \text{ for some } z \in \mathbb{R}\} \\
&= \{(x, y): z = x^2 \text{ and } y = z + 1 \text{ for some } z \in \mathbb{R}\} \\
&= \{(x, y): y = x^2 + 1\}. \\
S \circ R &= \{(x, y): (x, z) \in R \text{ and } (z, y) \in S \text{ for some } z \in \mathbb{R}\} \\
&= \{(x, y): z = x + 1 \text{ and } y = z^2 \text{ for some } z \in \mathbb{R}\} \\
&= \{(x, y): y = (x + 1)^2\}.
\end{aligned}$$

Clearly, $S \circ R \neq R \circ S$, since $x^2 + 1$ is seldom equal to $(x + 1)^2$.

The last theorem of this section presents several results about inversion, composition, and the identity relation. We prove part (b) and the first part of (c), leaving the rest as Exercise 10.

**Theorem 3.1.3** Suppose $A$, $B$, $C$, and $D$ are sets. Let $R$ be a relation from $A$ to $B$, $S$ be a relation from $B$ to $C$, and $T$ be a relation from $C$ to $D$.

**(a)** $(R^{-1})^{-1} = R$.
**(b)** $T \circ (S \circ R) = (T \circ S) \circ R$, so composition is associative.
**(c)** $I_B \circ R = R$ and $R \circ I_A = R$.
**(d)** $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof.

**(b)** The pair $(x, w) \in T \circ (S \circ R)$ for some $x \in A$ and $w \in D$
iff $(\exists z \in C)[(x, z) \in S \circ R$ and $(z, w) \in T]$
iff $(\exists z \in C)[(\exists y \in B)((x, y) \in R$ and $(y, z) \in S)$ and $(z, w) \in T]$
iff $(\exists z \in C)(\exists y \in B)[(x, y) \in R$ and $(y, z) \in S$ and $(z, w) \in T]$
iff $(\exists y \in B)(\exists z \in C)[(x, y) \in R$ and $(y, z) \in S$ and $(z, w) \in T]$

iff $(\exists y \in B)[(x, y) \in R$ and $(\exists z \in C)((y, z) \in S$ and $(z, w) \in T)]$
iff $(\exists y \in B)[(x, y) \in R$ and $(y, w) \in T \circ S]$
iff $(x, w) \in (T \circ S) \circ R$.
Therefore, $T \circ (S \circ R) = (T \circ S) \circ R$.

(c) ⟨*We first show that* $I_B \circ R \subseteq R$.⟩ Suppose $(x, y) \in I_B \circ R$. Then there exists $z \in B$ such that $(x, z) \in R$ and $(z, y) \in I_B$. Since $(z, y) \in I_B$, $z = y$. Thus $(x, y) \in R$ ⟨*since* $(x, y) = (x, z) \in R$⟩.
Conversely, suppose $(p, q) \in R$. Then $(q, q) \in I_B$ and thus $(p, q) \in I_B \circ R$. Thus $I_B \circ R = R$. ∎

The storage and manipulation of data in tables ($n$-tuple relations) is an important field of computer science called *relational databases*. Operations such as union and composition for ordered pairs may be extended to operations on $n$-tuples. One generalization of composition in relational databases is the "join" of two tables.

**Example.** Suppose the student information at a small university includes both directory information and billing information. We let $A$ be the set of first names, $B$ be last names, $C$ be 4-digit student ID numbers, $D$ be names of campus residence halls, $E$ be residence hall room numbers, $F$ be tuition amounts due, and $G$ be room charges due.

The student records in the directory may be described in a table $R$:

$R$ (directory)

| First Name | Last Name | Student ID | Residence Hall | Room Number |
|---|---|---|---|---|
| Krista | Maire | 1234 | Orlando | 77 |
| Harold | Dorman | 2490 | Mountain | 455 |
| Ferlin | Husky | 5555 | Dove | 213A |
| Martha | Reeves | 3215 | Vandella | 238 |
| Kim | Anen | 6920 | Bowie | 1979 |

The directory relation $R$ is a subset of $A \times B \times C \times D \times E$ consisting of five 5-tuples. The 5-tuple (Krista, Maire, 1234, Orlando, 77) is one student record in the directory $R$.

The financial information relation $S$ is a subset of $C \times F \times G$:

$S$ (financial)

| Student ID | Tuition | Room Charges |
|---|---|---|
| 1234 | \$80 | \$40 |
| 2490 | \$150 | \$20 |
| 5555 | \$75 | \$25 |
| 3215 | \$0 | \$0 |
| 6920 | \$0 | \$60 |

The **join** of these two tables, denoted $R \otimes S$, is a table with 7 columns. The rows of the table are obtained by merging 5-tuples from $R$ and 3-tuples from $S$ that share a common ID number:

$R \otimes S$

| First Name | Last Name | Student ID | Residence Hall | Room Number | Tuition | Room Charges |
|---|---|---|---|---|---|---|
| Krista | Maire | 1234 | Orlando | 77 | \$80 | \$40 |
| Harold | Dorman | 2490 | Mountain | 455 | \$150 | \$20 |
| Ferlin | Husky | 5555 | Dove | 213A | \$75 | \$25 |
| Martha | Reeves | 3215 | Vandella | 238 | \$0 | \$0 |
| Kim | Anen | 6920 | Bowie | 1979 | \$0 | \$60 |

The join operation is one of several database operations that allow a manager to create tables in response to requests for information (queries). There are many advantages to storing data in simple tables like $R$ and $S$, but requests such as "What is the room charge for Harold Dorman?" cannot be answered using either of the tables by itself.

## Exercises 3.1

1. Let $T$ be the relation $\{(3, 1), (2, 3), (3, 5), (2, 2), (1, 6), (2, 6), (1, 2)\}$. Find
   (a) Dom $(T)$. (b) Rng $(T)$.
   (c) $T^{-1}$. (d) $(T^{-1})^{-1}$.
2. Find the domain and range for the relation $W$ on $\mathbb{R}$ given by $x\ W\ y$ iff
   ★ (a) $y = 2x + 1$. (b) $y = x^2 + 3$.
   ★ (c) $y = \sqrt{x - 1}$. (d) $y = \dfrac{1}{x^2}$
   ★ (e) $y \le x^2$. (f) $|x| < 2$ and $y = 3$.
   (g) $|x| < 2$ or $y = 3$. (h) $y \ne x$.
3. Sketch the graph of each relation in Exercise 2.
4. The inverse of $R = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 2x + 1\}$ may be expressed in the form $R^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{x - 1}{2}\right\}$, the set of all pairs $(x, y)$ subject to some condition. Use this form to give the inverses of the following relations. In (i), (j), and (k), $P$ is the set of all people.
   ★ (a) $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x\}$
   (b) $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = -5x + 2\}$
   ★ (c) $R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = 7x - 10\}$
   (d) $R_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2 + 2\}$
   ★ (e) $R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = -4x^2 + 5\}$

(f) $R_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y < x + 1\}$
★ (g) $R_7 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y > 3x - 4\}$
(h) $R_8 = \left\{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \dfrac{2x}{x-2}\right\}$
★ (i) $R_9 = \{(x, y) \in P \times P: y \text{ is the father of } x\}$
(j) $R_{10} = \{(x, y) \in P \times P: y \text{ is a sibling of } x\}$
(k) $R_{11} = \{(x, y) \in P \times P: y \text{ loves } x\}$

5. Let $R = \{(1, 5), (2, 2), (3, 4), (5, 2)\}$, $S = \{(2, 4), (3, 4), (3, 1), (5, 5)\}$, and $T = \{(1, 4), (3, 5), (4, 1)\}$. Find
(a) $R \circ S$. ★ (b) $R \circ T$.
(c) $T \circ S$. ★ (d) $R \circ R$.
(e) $S \circ R$. (f) $T \circ T$.
(g) $R \circ (S \circ T)$. (h) $(R \circ S) \circ T$.

6. Find these composites for the relations defined in Exercise 4.
★ (a) $R_1 \circ R_1$ (b) $R_1 \circ R_2$
(c) $R_2 \circ R_2$ ★ (d) $R_2 \circ R_3$
(e) $R_2 \circ R_4$ (f) $R_4 \circ R_2$
★ (g) $R_4 \circ R_5$ (h) $R_6 \circ R_2$
(i) $R_6 \circ R_4$ ★ (j) $R_6 \circ R_6$
(k) $R_7 \circ R_7$ (l) $R_5 \circ R_5$
(m) $R_8 \circ R_8$ ★ (n) $R_3 \circ R_8$
(o) $R_8 \circ R_3$ ☆ (p) $R_9 \circ R_9$

7. Give the digraphs for these relations on the set $\{1, 2, 3\}$.
(a) $=$ (b) $S = \{(1, 3), (2, 1)\}$
(c) $\leq$ (d) $S^{-1}$, where $S = \{(1, 3), (2, 1)\}$
(e) $\neq$ (f) $S \circ S$, where $S = \{(1, 3), (2, 1)\}$

8. Let $A = \{a, b, c, d\}$. Give an example of relations $R$, $S$, and $T$ on $A$ such that
(a) $R \circ S \neq S \circ R$. (b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$.
(c) $S \circ R = T \circ R$ but $S \neq T$.
(d) $R$ and $S$ are nonempty, and $R \circ S$ and $S \circ R$ are empty.

9. Let $R$ be a relation from $A$ to $B$ and $S$ be a relation from $B$ to $C$.
(a) Prove that Dom $(S \circ R) \subseteq$ Dom $(R)$.
(b) Show by example that Dom $(S \circ R) =$ Dom $(R)$ may be false.
(c) Which of these two statements must be true:

$$\text{Rng}(S) \subseteq \text{Rng}(S \circ R) \quad \text{or} \quad \text{Rng}(S \circ R) \subseteq \text{Rng}(S)?$$

Give an example to show that the other statement may be false.

10. Complete the proof of Theorem 3.1.3.

11. Show by example that $(A \times B) \times C = A \times (B \times C)$ may be false.

12. Prove that if $A$ has $m$ elements and $B$ has $n$ elements, then there are $2^{mn}$ different relations from $A$ to $B$.

**13.** (a) Let $R$ be a relation from $A$ to $B$. For $a \in A$, define the **vertical section of $R$ at $a$** to be $R_a = \{y \in B: (a, y) \in R\}$. Prove that $\bigcup_{a \in A} R_a = \text{Rng}(R)$.

(b) Let $R$ be a relation from $A$ to $B$. For $a \in A$, define the **horizontal section of $R$ at $b$** to be ${}_bR = \{x \in A: (x, b) \in R\}$. Prove that $\bigcup_{b \in B} {}_bR = \text{Dom}(R)$.

**14.** We may define ordered triples in terms of ordered pairs by saying that $(a, b, c) = ((a, b), c)$. Use this definition to prove that $(a, b, c) = (x, y, z)$ iff $a = x$ and $b = y$ and $c = z$.

*Proofs to Grade*

**15.** Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

★ (a) **Claim.** $(A \times B) \cup C = (A \times C) \cup (B \times C)$.
***"Proof."*** $x \in (A \times B) \cup C$
iff $x \in A \times B$ or $x \in C$
iff $x \in A$ and $x \in B$ or $x \in C$
iff $x \in A \times C$ or $x \in B \times C$
iff $x \in (A \times C) \cup (B \times C)$. ■

★ (b) **Claim.** If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.
***"Proof."*** Suppose $A \times C \nsubseteq B \times D$. Then there exists $(a, c) \in A \times C$ with $(a, c) \notin B \times D$. But $(a, c) \in A \times C$ implies that $a \in A$ and $c \in C$, whereas $(a, c) \notin B \times D$ implies that $a \notin B$ and $c \notin D$. However, $A \subseteq B$ and $C \subseteq D$, so $a \in B$ and $c \in D$. This is a contradiction. Therefore, $A \times C \subseteq B \times D$. ■

(c) **Claim.** If $A \times B = A \times C$ and $A \neq \varnothing$, then $B = C$.
***"Proof."*** Suppose $A \times B = A \times C$. Then

$$\frac{A \times B}{A} = \frac{A \times C}{A}.$$

Therefore $B = C$. ■

★ (d) **Claim.** If $A \times B = A \times C$ and $A \neq \varnothing$, then $B = C$.
***"Proof."*** To show $B = C$, suppose $b \in B$. Choose any $a \in A$. Then $(a, b) \in A \times B$. But since $A \times B = A \times C$, $(a, b) \in A \times C$. Thus $b \in C$. This proves $B \subseteq C$. A proof of $C \subseteq B$ is similar. Therefore, $B = C$. ■

(e) **Claim.** Let $R$ and $S$ be relations from $A$ to $B$ and from $B$ to $C$, respectively. Then $S \circ R = (R \circ S)^{-1}$.
***"Proof."*** The pair $(x, y) \in S \circ R$ iff $(y, x) \in R \circ S$ iff $(x, y) \in (R \circ S)^{-1}$ Therefore, $S \circ R = (R \circ S)^{-1}$. ■

(f) **Claim.** Let $R$ be a relation from $A$ to $B$. Then $I_A \subseteq R^{-1} \circ R$.
***"Proof."*** Suppose $(x, x) \in I_A$. Choose any $y \in B$ such that $(x, y) \in R$. Then, $(y, x) \in R^{-1}$. Thus $(x, x) \in R^{-1} \circ R$. Therefore, $I_A \subseteq R^{-1} \circ R$. ■

(g) **Claim.** Suppose $R$ is a relation from $A$ to $B$. Then $R^{-1} \circ R \subseteq I_A$.
***"Proof."*** Let $(x, y) \in R^{-1} \circ R$. Then for some $z \in B$, $(x, z) \in R$ and $(z, y) \in R^{-1}$. Thus $(y, z) \in R$. Since $(x, z) \in R$ and $(y, z) \in R$, $x = y$. Thus $(x, y) = (x, x)$ and $x \in A$, so $(x, y) \in I_A$. ■

## 3.2 Equivalence Relations

The goal of this section is to describe a way to equate objects in a set according to some value, property, or meaning. We might say that among all students who completed a certain math class, students are equivalent if they had the same numeric score on the final exam. With this meaning of equivalence, a student with a score of 87 on the final exam is related to every other student with a score of 87 and not related to any other student. We could also have said that two students are equivalent if they have the same favorite movie, or if they have the same blood type.

The three properties we define next, when taken together, comprise what we mean by objects being equivalent.

> DEFINITIONS Let $A$ be a set and $R$ be a relation on $A$.
>
> $R$ is **reflexive on $A$** iff for all $x \in A$, $x\,R\,x$.
> $R$ is **symmetric** iff for all $x$ and $y \in A$, if $x\,R\,y$, then $y\,R\,x$.
> $R$ is **transitive** iff for all $x$, $y$, and $z \in A$, if $x\,R\,y$ and $y\,R\,z$, then $x\,R\,z$.

The relation $R$, defined as "had the same final exam score," on the set $C$ of all students in a given class has all three of these properties. $R$ is symmetric because if student $x$ had the same score as student $y$, then student $y$ must have had the same score as student $x$. $R$ is transitive because if student $x$ had the same score as student $y$ and student $y$ had the same score as student $z$, then $x$ had the same score as $z$. Finally, for every student $x$ in $C$, $x$ must have had the same score as $x$. Thus $R$ is reflexive on $C$.

To prove that a relation $R$ is symmetric or transitive, we usually give a direct proof, because these properties are defined by conditional sentences. A proof that $R$ is reflexive on $A$ is different. What we must do is show that for all $x \in A$, $x$ is $R$-related to $x$.

For a relation $R$ on a nonempty set $A$, only the reflexive property actually asserts that some ordered pairs belong to $R$. The empty relation $\varnothing$ is not reflexive on a set $A$ except in the special case when $A$ is the empty set. The empty relation $\varnothing$ is, however, symmetric and transitive for any set $A$. See Exercise 4. For each of the three properties there is an alternate condition (involving the identity relation or the operations of inversion or composition) that may be used to prove that a relation has or does not have that property. See Exercise 13.

To prove that a relation $R$ on a set $A$ is not reflexive on $A$, we must show that there exists some $x \in A$ such that $x \not R\, x$. Since the denial of "If $x\,R\,y$ then $y\,R\,x$" is "$x\,R\,y$ and not $y\,R\,x$," a relation $R$ is not symmetric iff there are elements $x$ and $y$ in $A$ such that $x\,R\,y$ and $y \not R\, x$. Likewise, $R$ is not transitive iff there exist elements $x$, $y$, and $z$ in $A$ such that $x\,R\,y$ and $y\,R\,z$ but $x \not R\, z$.

Examples. For $B = \{2, 5, 6, 7\}$, let $S = \{(2, 5), (5, 6), (2, 6), (7, 7)\}$ and $T = \{(2, 6), (5, 6)\}$. Since $6 \not S\, 6$ and $2 \not T\, 2$, neither $S$ nor $T$ is reflexive on $B$. The relation $S$ is not symmetric because $2\, S\, 5$, but $5 \not S\, 2$. Likewise, $T$ is not symmetric because $5\, T\, 6$ but $6 \not T\, 5$.

Both $S$ and $T$ are transitive relations. To verify that $S$ is transitive we check all pairs $(x, y)$ in $S$ with all pairs of the form $(y, z)$. We have $(2, 5)$ and $(5, 6)$ in $S$, so we must have $(2, 6)$; we have $(7, 7)$ and $(7, 7)$ in $S$ so we must have $(7, 7)$. The relation $T$ is transitive for a different reason: there do not exist $x, y, z$ in $B$ such that $x\, T\, y$ and $y\, T\, z$. Because its antecedent is false, the conditional sentence "If $x\, T\, y$ and $y\, T\, z$, then $x\, T\, z$" is true.

Example. Let $R$ be the relation "is a subset of" on $\mathcal{P}(\mathbb{Z})$, the power set of $\mathbb{Z}$. $R$ is reflexive on $\mathcal{P}(\mathbb{Z})$ since every set is a subset of itself. $R$ is transitive by Theorem 2.1.1(c). Notice that $\{1, 2\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\} \not\subseteq \{1, 2\}$. Therefore, $R$ is not symmetric.

Example. Let STNR designate the relation $\{(x, y) \in \mathbb{Z} \times \mathbb{Z}\colon xy > 0\}$ on $\mathbb{Z}$. In this example, $x$ STNR $x$ for all $x$ in $\mathbb{Z}$ except the integer 0; hence the relation STNR is not reflexive on $\mathbb{Z}$. STNR is symmetric since, if $x$ and $y$ are integers and $xy > 0$, then $yx > 0$. STNR is also transitive. To verify this, we assume that $x$ STNR $y$ and $y$ STNR $z$. Then $xy > 0$ and $yz > 0$. If $y$ is positive, then both $x$ and $z$ are positive; so $xz > 0$. If $y$ is negative, then both $x$ and $z$ are negative; so $xz > 0$. Thus in either case, $x$ STNR $z$. This relation gets its name from the fact that it is symmetric, transitive, and not reflexive on $\mathbb{Z}$.

For a relation $R$ on a set $A$, the properties of reflexivity on $A$, symmetry, and transitivity can also be characterized by properties in the digraph of $R$:

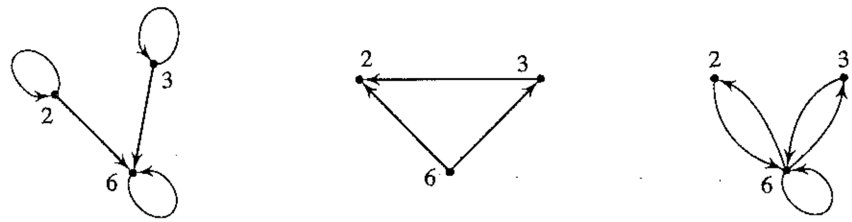
> $R$ is reflexive on $A$ iff every vertex of the digraph has a loop.
> $R$ is symmetric iff between any two vertices there are either no edges or an edge in both directions.
> $R$ is transitive iff whenever there is an edge from vertex $x$ to $y$ and an edge from vertex $y$ to $z$, there is an edge (a direct route) from $x$ to $z$.

Examples. Figure 3.2.1 shows the digraphs of three relations on $A = \{2, 3, 6\}$. Figure 3.2.1(a) is the digraph of the relation "divides" and Figure 3.2.1(b) is the digraph of "$>$." Figure 3.2.1(c) is the digraph of the relation $S$, where $x\, S\, y$ iff $x + y > 7$.

There is a loop at every vertex in Figure 3.2.1(a) because the relation "divides" is reflexive: Every integer divides itself. The relations "$>$" and $S$ are not reflexive; there is no loop at 2 in Figure 3.2.1(b) or (c).

$S$ is a symmetric relation, but the others are not. In Figure 3.2.1(a) there is an arc from 2 to 6, but not in the reverse direction; in Figure 3.2.1(b) there is an arc from 6 to 2, but not from 2 to 6.

The relation $S$ is not transitive—there is an arc from 2 to 6 and one from 6 to 3, but no arc from 2 to 3. The other two relations are transitive. Note that for the digraph in Figure 3.2.1(a), every pair of arcs to be checked for transitivity involves a loop. For example, there is an arc from 3 to 3 and an arc from 3 to 6; the shortcut is to go directly from 3 to 6.




(a) divides (b) > (c) $S$

**Figure 3.2.1**

For every set $A$, the identity relation $I_A$ is reflexive on $A$, symmetric, and transitive. The identity relation is, in fact, the relation "equals," because $x \, I_A \, y$ iff $x = y$. Equality is a way of comparing objects according to whether they are the same. Equivalence relations, defined next, are a means for relating objects according to whether they are, if not identical, at least alike in the sense that they share a common trait. For example, if $T$ is the set of all triangles, we might say two triangles are "the same" (equivalent) when they are congruent. This generates the relation $R = \{(x, y) \in T \times T: x \text{ is congruent to } y\}$ on $T$, which is reflexive on $T$, symmetric, and transitive. The notion of equivalence, then, is embodied in these three properties.

> DEFINITION A relation $R$ on a set $A$ is an **equivalence relation on $A$** iff $R$ is reflexive on $A$, symmetric, and transitive.

Suppose we say two integers are related iff they have the same parity. For this relation, $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x + y \text{ is even}\}$, we see that all the odd integers are related to one another (since the sum of two odd numbers is even) and all the evens are related to each other. The relation $R$ is reflexive on $\mathbb{Z}$, symmetric, and transitive and is, therefore, an equivalence relation.

For the set $P$ of all people, let $L$ be the relation on $P$ given by $x \, L \, y$ iff $x$ and $y$ have the same family name. We have Lucy Brown $L$ Charlie Brown, James Madison $L$ Dolly Madison, and so on. If we make the assumption that everyone has exactly one family name, then $L$ is an equivalence relation on $P$.

The subset of $P$ consisting of all people who are $L$-related to Charlie Brown is the set of all people whose family name is Brown. This set contains Charlie by reflexivity. It also contains Sally Brown, James Brown, Buster Brown, Leroy Brown, and all other people who are like Charlie Brown in the sense that they have Brown as a family name. The same is true for the Madisons: The set of people $L$-related to Dolly Madison is the set of all people with the family name Madison.

DEFINITIONS Let $R$ be an equivalence relation on a set $A$. For $x \in A$, the **equivalence class** of $x$ determined by $R$ is the set

$$x/R = \{y \in A: x\,R\,y\}.$$

When $R$ is fixed throughout a discussion or clear from the context, the notations $[x]$ and $\overline{x}$ are commonly used instead of $x/R$.

We read $x/R$ as "the class of $x$ modulo $R$," or simply "$x$ mod $R$."

The set $A/R = \{x/R: x \in A\}$ of all equivalence classes is called **$A$ modulo $R$**.

The equivalence class of Charlie Brown modulo $L$ is the set of all people whose family name is Brown. Furthermore, Buster Brown/$L$ is the same set as Charlie Brown/$L$.

Example. The relation $H = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$ is an equivalence relation on the set $A = \{1, 2, 3\}$. Here $1/H = 2/H = \{1, 2\}$ and $3/H = \{3\}$. Thus $A/H = \{\{1, 2\}, \{3\}\}$.

Example. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x^2 = y^2\}$. $S$ is an equivalence relation on $\mathbb{R}$. We have $\overline{2} = \{2, -2\}$, $\overline{\pi} = \{-\pi, \pi\}$, etc. Also, $\overline{0} = \{0\}$. In this example, for every $x \in \mathbb{R}$ the equivalence class of $x$ and the equivalence class of $-x$ are the same. $\mathbb{R}$ modulo $S$ is $\mathbb{R}/S = \{\{x, -x\}: x \in \mathbb{R}\}$.

Example. For the equivalence relation $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x + y \text{ is even}\}$ on $\mathbb{Z}$, there are only two equivalence classes: $D$, the set of all odd integers and $E$, the set of even integers. Thus $\mathbb{Z}/R = \{D, E\}$.

Note that in the examples above—$A/H$, $\mathbb{R}/S$, and $\mathbb{Z}/R$—any two equivalence classes are either equal or disjoint. The next theorem tells us for all equivalence relations, distinct equivalence classes never "overlap."

Theorem 3.2.1 Let $R$ be an equivalence relation on a nonempty set $A$. For all $x$, $y$ in $A$,

**(a)** $x/R \subseteq A$ and $x \in x/R$. Thus every equivalence class is a nonempty subset of $A$.

**(b)** $x\,R\,y$ iff $x/R = y/R$. Thus elements of $A$ are related iff their equivalence classes are identical.

**(c)** $x \not R\, y$ iff $x/R \cap y/R = \varnothing$. Thus elements of $A$ are unrelated iff their equivalence classes are disjoint.

Proof.

(a) By the definition of $x/R$, $x/R \subseteq A$. Since $R$ is reflexive on $A$, $x\,R\,x$. Thus $x \in x/R$.

(b) (i) Suppose $x\,R\,y$. To show $x/R = y/R$, we first show $x/R \subseteq y/R$. Let $z \in x/R$. Then $x\,R\,z$. From $x\,R\,y$, by symmetry, $y\,R\,x$. Then, by transitivity, $y\,R\,z$. Thus $z \in y/R$. The proof that $y/R \subseteq x/R$ is similar.

(ii) Suppose $x/R = y/R$. Since $y \in y/R$, $y \in x/R$. Thus $x\,R\,y$.

(c) (i) If $x/R \cap y/R = \varnothing$, then, since $y \in y/R$, $y \notin x/R$. Thus $x \not R\, y$.

(ii) Finally, we show $x \not R\, y$ implies $x/R \cap y/R = \varnothing$. ⟨*We prove the contrapositive.*⟩ Suppose $x/R \cap y/R \neq \varnothing$. Let $k \in x/R \cap y/R$. Then $x\,R\,k$ and $y\,R\,k$. Therefore, $x\,R\,k$ and $k\,R\,y$. Thus $x\,R\,y$. ■

For the rest of this section, we explore the properties of an equivalence relation that has a multitude of important applications. This relation, called congruence, provides a valuable way to deal with questions associated with divisibility in the integers. The notion of congruence, first introduced by Carl Friedrich Gauss,* leads to modular arithmetic, which is an abstraction of our usual arithmetic, and this leads in turn to methods for converting computational problems with large integers into more manageable problems.

> DEFINITIONS Let $m$ be a fixed positive integer. For $x, y \in \mathbb{Z}$, we say **$x$ is congruent to $y$ modulo $m$** iff $m$ divides $(x - y)$. We write $\boldsymbol{x \equiv_m y}$, or simply $\boldsymbol{x \equiv y \ (\mathrm{mod}\ m)}$. The number $m$ is called the **modulus** of the congruence.

Examples. Using 3 as the modulus, $4 \equiv 1 \pmod 3$ because 3 divides $4 - 1$. Likewise, $10 \equiv 16 \pmod 3$ because 3 divides $10 - 16 = -6$. Since 3 does not divide $5 - (-9) = 14$, we have $5 \not\equiv -9 \pmod 3$. It is easy to see that 0 is congruent to $0, 3, -3, 6$, and $-6$ and, in fact, 0 is congruent modulo 3 to every multiple of 3.

**Theorem 3.2.2** For every fixed positive integer $m$, $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.

Proof. We note that $\equiv_m$ is a set of ordered pairs of integers and, hence, is a relation on $\mathbb{Z}$. ⟨*Now we show that $\equiv_m$ is reflexive on $\mathbb{Z}$, symmetric, and transitive.*⟩

(i) To show reflexivity on $\mathbb{Z}$, let $x$ be an integer. We show that $x \equiv x \pmod m$. Since $m \cdot 0 = 0 = x - x$, $m$ divides $x - x$. Thus $\equiv_m$ is reflexive on $\mathbb{Z}$.

(ii) For symmetry, suppose $x \equiv y \pmod m$. Then $m$ divides $x - y$. Thus there is an integer $k$ so that $x - y = km$. But this means that $-(x - y) = -(km)$, or that $y - x = (-k)m$. Therefore, $m$ divides $y - x$, so that $y \equiv x \pmod m$.

* The German Carl Friedrich Gauss (1777–1855), one of the greatest mathematicians of all time, also made major contributions to astronomy and physics. Congruence and modular arithmetic (and much more) appeared in his masterwork *Disquisitiones Arithmeticae*, which he completed at the age of 21. He proved the Fundamental Theorem of Algebra and the Prime Number Theorem, among many other results in number theory, statistics, analysis, and differential geometry.

**(iii)** Suppose $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$. Thus $m$ divides both $x - y$ and $y - z$. Therefore, there exist integers $h$ and $k$ such that $x - y = hm$ and $y - z = km$. But then $h + k$ is an integer, and

$$x - z = (x - y) + (y - z) = hm + km = (h + k)m.$$

Thus $m$ divides $x - z$, so $x \equiv z \pmod{m}$. Therefore, $\equiv_m$ is transitive. ∎

> DEFINITION The set of equivalence classes for the relation $\equiv_m$ is denoted $\mathbb{Z}_m$.

We can now determine the set $\mathbb{Z}_3$ of all equivalence classes modulo 3. For $x \in \mathbb{Z}$, the equivalence class of $x$ is $\{y \in \mathbb{Z}\colon x \equiv_3 y\}$, which we now denote by $\bar{x}$. Since the integers congruent to 0 (mod 3) are exactly the multiples of 3, we have

$$\overline{0} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}.$$

To form the equivalence class of 1, denoted $\overline{1}$, we begin with 1 (because $1 \equiv_3 1$) and repeatedly add or subtract 3. This produces the positive integers 4, 7, 10, 13, ... and the negative integers −2, −5, −8, ... that are congruent to 1 modulo 3, so

$$\overline{1} = \{\ldots, -8, -5, -2, 1, 4, 7, 10, 13, \ldots\}.$$

In the same way we form

$$\overline{2} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}.$$

If we compute $\overline{3} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$ we find that $\overline{3} = \overline{0}$ and in fact $\overline{4} = \overline{1}, \overline{5} = \overline{2}, \overline{6} = \overline{0}$, etc., so there are really only three different equivalence classes. We have found that $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$.

Notice that the class of 0 modulo 3 above is not the same as the congruence class of 0 modulo 4. The class of 0 modulo 4 contains 0, ±4, ±8, ±12, and all the other multiples of 4. See Exercise 9.

Using the notation $\bar{x}$ for the equivalence class of $x$ modulo $m$ works well as long as the modulus remains unchanged, but suppose we want to compare computations with two different moduli. To work with elements of, say, $\mathbb{Z}_6$ as well as elements of $\mathbb{Z}_3$, we will write elements of $\mathbb{Z}_6$ as [0], [1], [2], [3], [4], and [5], to distinguish them from the elements $\overline{0}$, $\overline{1}$, and $\overline{2}$, of $\mathbb{Z}_3$.

The 12 hours on the clock correspond to the 12 classes in $\mathbb{Z}_{12}$. Rather than talking about hours beyond 12 o'clock, we start over again with 1 o'clock instead of 13 o'clock because $13 \equiv 1 \pmod{12}$, and 2 o'clock instead of 14 o'clock because $14 \equiv 2 \pmod{12}$, etc. The hours on a clock face show only the hours since the previous midnight or noon. We are so accustomed to working with equivalence classes modulo 12 that we routinely do arithmetic with them: 9 hours after 8 o'clock is 5 o'clock, because $8 + 9 = 17$ and $17 \equiv 5$ (modulo 12) and 4 hours before 3 o'clock is 11 o'clock, because $3 - 4 = -1 \equiv 11$ (modulo 12).

Our next theorem will show that there are always $m$ different equivalence classes for the relation $\equiv_m$ and the set $\mathbb{Z}_m$ is always $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$. It is helpful to observe that $0, 1, 2, \ldots,$ and $m - 1$ are exactly all the possible remainders when integers are divided by $m$. For this reason the elements of $\mathbb{Z}_m$ are sometimes called the *residue* (or remainder) classes modulo $m$.

**Theorem 3.2.3** Let $m$ be a fixed positive integer. Then

(a) For integers $x$ and $y$, $x \equiv y \pmod m$ iff the remainder when $x$ is divided by $m$ equals the remainder when $y$ is divided by $m$.

(b) $\mathbb{Z}_m$ consists of $m$ distinct equivalence classes: $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.

**Proof.**

(a) Let $x$ and $y$ be integers. By the Division Algorithm, there exist integers $q$, $r$, $t$, and $s$ such that $x = mq + r$, with $0 \le r < m$ and $y = mt + s$, with $0 \le s < m$. (*We must show that* $x \equiv y \pmod m$ *iff* $r = s$.) Then

$$
\begin{aligned}
x \equiv y \pmod m \text{ iff } & m \text{ divides } x - y \\
\text{iff } & m \text{ divides } (mq + r) - (mt + s) \\
\text{iff } & m \text{ divides } m(q - t) + (r - s) \\
\text{iff } & m \text{ divides } r - s \\
\text{iff } & r = s. \text{ (This is because } 0 \le r < m \text{ and } 0 \le s < m.)
\end{aligned}
$$

(b) (*We first show that* $\mathbb{Z}_m = \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.) For each $k$, where $0 \le k \le m - 1$, the set $\overline{k}$ is an equivalence class, so $\{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$ is a subset of $\mathbb{Z}_m$. Now suppose $\overline{x} \in \mathbb{Z}_m$ for some integer $x$. By the Division Algorithm, there exist integers $q$ and $r$ such that $x = mq + r$, with $0 \le r < m$. Then $x - r = mq$, so $m$ divides $x - r$. Thus $x \equiv r \pmod m$. By Theorem 3.2.1(b) $\overline{x} = \overline{r}$. Therefore $\mathbb{Z}_m \subseteq \{\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}\}$.

Finally we will know that $\mathbb{Z}_m$ has exactly $m$ elements when we show that the equivalence classes $\overline{0}, \overline{1}, \overline{2}, \ldots, \overline{m-1}$ are all distinct. Suppose $\overline{k} = \overline{r}$ where $0 \le r \le k \le m - 1$. Then $k \equiv r \pmod m$, and thus $m$ divides $k - r$. But $0 \le k - r \le m - 1$, so $k - r = 0$. Then $k = r$. Therefore the $m$ equivalence classes are distinct. ∎

## Exercises 3.2

1. Indicate which of the following relations on the given sets are reflexive on a given set, which are symmetric, and which are transitive.

★ (a) $\{(1, 2)\}$ on $\{1, 2\}$ (b) $\le$ on $\mathbb{N}$

(c) $=$ on $\mathbb{N}$ (d) $<$ on $\mathbb{N}$

★ (e) $\geq$ on $\mathbb{N}$
(f) $\neq$ on $\mathbb{N}$
(g) "divides" on $\mathbb{N}$
(h) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x + y = 10\}$
(i) $\{(1, 5), (5, 1), (1, 1)\}$ on the set $A = \{1, 2, 3, 4, 5\}$
(j) $\perp = \{(l, m): l$ and $m$ are lines and $l$ is perpendicular to $m\}$ on the set of all lines in a plane
(k) $R$, where $(x, y)\, R\, (z, w)$ iff $x + z \leq y + w$, on the set $\mathbb{R} \times \mathbb{R}$
★ (l) $S$, where $x\, S\, y$ iff $x$ is a sibling of $y$, on the set $P$ of all people
(m) $T$, where $(x, y)\, T\, (z, w)$ iff $x + y \leq z + w$, on the set $\mathbb{R} \times \mathbb{R}$

2. Let $A = \{1, 2, 3\}$. List the ordered pairs and draw the digraph of a relation on $A$ with the given properties.
★ (a) not reflexive, not symmetric, and not transitive
(b) reflexive, not symmetric, and not transitive
(c) not reflexive, symmetric, and not transitive
★ (d) reflexive, symmetric, and not transitive
(e) not reflexive, not symmetric, and transitive
(f) reflexive, not symmetric, and transitive
(g) not reflexive, symmetric, and transitive
(h) reflexive, symmetric, and transitive

3. For each part of Exercise 2, give an example of a relation on $\mathbb{R}$ with the desired properties.

4. Let $R$ be a relation on a set $A$. Prove that
(a) if $A$ is nonempty, the empty relation $\varnothing$ is not reflexive on $A$.
(b) the empty relation $\varnothing$ is symmetric and transitive for every set $A$.

5. For each of the following, prove that the relation is an equivalence relation. Then give information about the equivalence classes as specified.
(a) The relation $R$ on $\mathbb{R}$ given by $x\, R\, y$ iff $x - y \in \mathbb{Q}$. Give the equivalence class of 0; of 1, of $\sqrt{2}$.
(b) The relation $R$ on $\mathbb{N}$ given by $m\, R\, n$ iff $m$ and $n$ have the same digit in the tens places. Find an element of $106/R$ that is less than 50; between 150 and 300; greater than 1,000. Find three such elements in the equivalence class $635/R$.
(c) The relation $V$ on $\mathbb{R}$ given by $x\, V\, y$ iff $x = y$ or $xy = 1$. Give the equivalence class of 3; of $-\frac{2}{3}$; of 0.
★ (d) On $\mathbb{N}$, the relation $R$ given by $a\, R\, b$ iff the prime factorizations of $a$ and $b$ have the same number of 2's. For example, $16\, R\, 80$ because $16 = 2^4$ and $80 = 2^4 \cdot 5$. Name three elements in each of these classes: $1/R$, $4/R$, $72/R$.
(e) The relation $T$ on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\, T\, (a, b)$ iff $x^2 + y^2 = a^2 + b^2$. Sketch the equivalence class of $(1, 2)$; of $(4, 0)$.
(f) For the set $X = \{m, n, p, q, r, s\}$, let $R$ be the relation on $\mathcal{P}(X)$ given by $A\, R\, B$ iff $A$ and $B$ have the same number of elements. List all the elements in $\{m\}/R$; in $\{m, n, p, q, r, s\}/R$. How many elements are in $X/R$? How many elements are in $\mathcal{P}(X)/R$?

(g) The relation $P$ on $\mathbb{R} \times \mathbb{R}$ defined by $(x, y)\, P\, (z, w)$ iff $|x - y| = |z - w|$. Name at least one ordered pair in each quadrant that is related to $(3, 0)$. Describe all ordered pairs in the equivalence class of $(0, 0)$; in the class of $(1, 0)$.

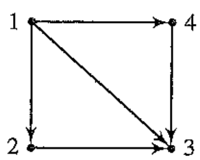
(h) Let $R$ be the relation on the set of all differentiable functions defined by $f\, R\, g$ iff $f$ and $g$ have the same first derivative, that is, $f' = g'$. Name three elements in each of these classes: $x^2/R$, $(4x^3 + 10x)/R$. Describe $x^3/R$ and $7/R$.

(i) The relation $T$ on $\mathbb{R}$ given by $x\, T\, y$ iff $\sin x = \sin y$. Describe the equivalence class of 0; of $\pi/2$; of $\pi/4$.

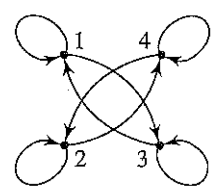
**6.** Let $R$ be the relation on $\mathbb{Q}$ defined by $\frac{p}{q}\, R\, \frac{s}{t}$ iff $pt = qs$. Show that $R$ is an equivalence relation. Describe all ordered pairs in the equivalence class of $\frac{2}{3}$.

**7.** Which of these digraphs represent relations that are (i) reflexive? (ii) symmetric? (iii) transitive?

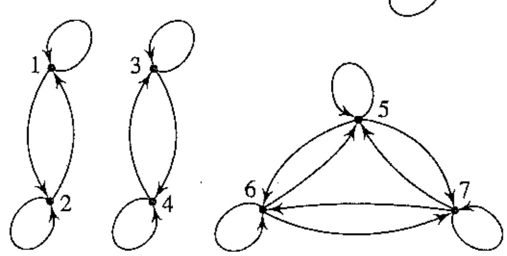
★ (a)



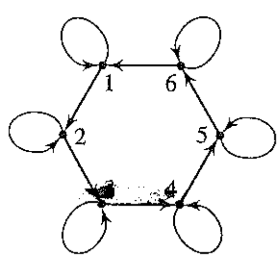

(b)




★ (c)




(d)




**8.** Determine the equivalence classes for the relation of

★ (a) congruence modulo 5.
(b) congruence modulo 8.
(c) congruence modulo 1.
(d) congruence modulo 7.

**9.** Name a positive integer and a negative integer that are

(a) congruent to 0 (mod 5) and not congruent to 0 (mod 6).
(b) congruent to 0 (mod 5) and congruent to 0 (mod 6).
(c) congruent to 2 (mod 4) and congruent to 8 (mod 6).
(d) congruent to 3 (mod 4) and congruent to 3 (mod 5).
(e) congruent to 1 (mod 3) and congruent to 1 (mod 7).

10. Using the fact that $\equiv_m$ is an equivalence relation on $\mathbb{Z}$ and without reference to Theorems 3.2.1 and 3.2.3, prove that for all $x$ and $y$ in $\mathbb{Z}$:
    (a) $x \in \bar{x}$.
    (b) $\bar{x} \neq \varnothing$.
    (c) if $x \equiv_m y$, then $\bar{x} = \bar{y}$.
    (d) if $\bar{x} = \bar{y}$, then $x \equiv_m y$.
    (e) if $\bar{x} \cap \bar{y} \neq \varnothing$, then $\bar{x} = \bar{y}$.
    (f) if $\bar{x} \cap \bar{y} = \varnothing$, then $\bar{x} \neq \bar{y}$.

11. Consider the relation $S$ on $\mathbb{N}$ defined by $x\, S\, y$ iff 3 divides $x + y$. Prove that $S$ is not an equivalence relation.

12. Suppose that $R$ and $S$ are equivalence relations on a set $A$. Prove that $R \cap S$ is an equivalence relation on $A$.

13. The properties of reflexivity, symmetry, and transitivity are related to the identity relation and the operations of inversion and composition. Prove that
    (a) $R$ is a reflexive relation on $A$ iff $I_A \subseteq R$.
    ★ (b) $R$ is symmetric iff $R = R^{-1}$.
    (c) $R$ is transitive iff $R \circ R \subseteq R$.

14. Prove that if $R$ is a symmetric, transitive relation on $A$ and the domain of $R$ is $A$, then $R$ is reflexive on $A$.

15. Let $R$ be a relation on the set $A$.
    ★ (a) Prove that $R \cup R^{-1}$ is symmetric. ($R \cup R^{-1}$ is the **symmetric closure** of $R$.)
    (b) Prove that if $S$ is a symmetric relation on $A$ and $R \subseteq S$, then $R^{-1} \subseteq S$.

16. Let $R$ be a relation on the set $A$. Define $T_R = \{(x, y) \in A \times A$: for some $n \in \mathbb{N}$ there exists $a_0 = x, a_1, a_2, \ldots, a_n = y \in A$ such that $(a_0, a_1), (a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n) \in R\}$.
    (a) Prove that $T_R$ is transitive. ($T_R$ is the **transitive closure** of $R$.)
    (b) Prove that if $S$ is a transitive relation on $A$ and $R \subseteq S$, then $T_R \subseteq S$.

17. The **complement** of a digraph has the same vertex set as the original digraph, and an arc from $x$ to $y$ exactly when the original digraph does not have an arc from $x$ to $y$. The two digraphs shown below are complementary. Call a digraph symmetric (transitive) iff its relation is symmetric (transitive).


    (a) Show that the complement of a symmetric digraph is symmetric.
    (b) Show by example that the complement of a transitive digraph need not be transitive.

18. Let $L$ be a relation on a set $A$ that is reflexive on $A$ and transitive but not necessarily symmetric. Let $R$ be the relation defined on $A$ by $x\, R\, y$ iff $x\, L\, y$ and $y\, L\, x$. Prove that $R$ is an equivalence relation.

***Proofs to Grade***

19. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.

(a) **Claim.** If the relation $R$ is symmetric and transitive, it is also reflexive.
***"Proof."*** Since $R$ is symmetric, if $(x, y) \in R$, then $(y, x) \in R$. Thus $(x, y) \in R$ and $(y, x) \in R$, and since $R$ is transitive, $(x, x) \in R$. Therefore, $R$ is reflexive. ■

(b) **Claim.** The relation $T$ on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\, T\, (r, s)$ iff $x + y = r + s$ is symmetric.
***"Proof."*** Suppose $(x, y) \in \mathbb{R} \times \mathbb{R}$. Then $(x, y)\, T\, (y, x)$ because $x + y = y + x$. Therefore, $T$ is symmetric. ■

(c) **Claim.** The relation $W$ on $\mathbb{R} \times \mathbb{R}$ given by $(x, y)\, W\, (r, s)$ iff $x - r = y - s$ is symmetric.
***"Proof."*** Suppose $(x, y)$ and $(r, s)$ are in $\mathbb{R} \times \mathbb{R}$ and $(x, y)\, W\, (r, s)$. Then $x - r = y - s$. Therefore, $r - x = s - y$, so $(r, s)\, W\, (x, y)$. Thus $W$ is symmetric. ■

(d) **Claim.** If the relations $R$ and $S$ are symmetric, then $R \cap S$ is symmetric.
***"Proof."*** Let $R$ be the relation of congruence modulo 10 and $S$ the relation of congruence modulo 6 on the integers. Both $R$ and $S$ are symmetric. If $(x, y) \in R \cap S$, then 6 and 10 divide $x - y$. Therefore, 2, 3, and 5 all divide $x - y$, so 30 divides $x - y$. Also if 30 divides $x - y$, then 6 and 10 divide $x - y$, so $R \cap S$ is the relation of congruence modulo 30. Therefore, $R \cap S$ is symmetric. ■

(e) **Claim.** If the relations $R$ and $S$ are symmetric, then $R \cap S$ is symmetric.
***"Proof."*** Suppose $(x, y) \in R \cap S$. Then $(x, y) \in R$ and $(x, y) \in S$. Since $R$ and $S$ are symmetric, $(y, x) \in R$ and $(y, x) \in S$. Therefore, $(y, x) \in R \cap S$. ■

★ (f) **Claim.** If the relations $R$ and $S$ are transitive, then $R \cap S$ is transitive.
***"Proof."*** Suppose $(x, y) \in R \cap S$ and $(y, z) \in R \cap S$. Then $(x, y) \in R$ and $(y, z) \in S$. Therefore, $(x, z) \in R \cap S$. ■

## 3.3 Partitions

Partitioning is frequently used to organize the world around us. The United States, for example, is partitioned in several ways—by postal zip codes, state boundaries, time zones, etc. In each case nonempty subsets of the United States are defined that do not overlap and that together comprise the entire country. This section introduces this concept of partitioning of a set and describes the close relationship between partitions and equivalence relations.

DEFINITION Let $A$ be a nonempty set. $\mathcal{P}$ is a **partition of $A$** iff $\mathcal{P}$ is a set of subsets of $A$ such that

**(i)** If $X \in \mathcal{P}$, then $X \neq \varnothing$.
**(ii)** If $X \in \mathcal{P}$ and $Y \in \mathcal{P}$, then $X = Y$ or $X \cap Y = \varnothing$.
**(iii)** $\bigcup_{X \in \mathcal{P}} X = A$.

The set $W$ of all employees in a large work area can be partitioned into work groups by putting up physical partitions (walls) to form cubicles. If we are careful so that (i) every cubicle contains at least one worker, (ii) no worker is assigned to two different cubicles, and (iii) every worker must be in some cubicle, then we have formed a partition of $W$. Notice that the workers are not elements of the partition; each element of the partition is a set of workers within a common cubicle. In Figure 3.3.1, $W$ is a set of 6 workers and the partition of $W$ consists of four sets—two sets each with two workers and two sets each with a single worker.

**Figure 3.3.1**

Examples. The 2-element family $\mathcal{P} = \{E, D\}$, where $E$ is the even integers and $D$ is the odd integers, is a partition of $\mathbb{Z}$. The 3-element collection $\mathcal{H} = \{\mathbb{N}, \{0\}, \mathbb{Z}^-\}$, where $\mathbb{Z}^-$ is the set of negative integers is also a partition of $\mathbb{Z}$. For each $k \in \mathbb{Z}$, let $A_k = \{3k, 3k+1, 3k+2\}$. The family $\mathcal{T} = \{A_k : k \in \mathbb{Z}\}$ is an infinite family that is a partition of $\mathbb{Z}$. Some elements of $\mathcal{T}$ are $A_0 = \{0, 1, 2\}$, $A_1 = \{3, 4, 5\}$, and $A_{-1} = \{-3, -2, -1\}$.

Two other partitions of $\mathbb{Z}$ are $\{\ldots, \{-3\}, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \ldots\}$ and $\{\mathbb{Z}\}$. In fact, for any nonempty set $A$, the families $\{\{x\}: x \in A\}$ and $\{A\}$ are partitions of $A$.

Example. For each $n \in \mathbb{Z}$, let $G_n = [n, n+1)$. The collection $\{G_n: n \in \mathbb{Z}\}$ of half open intervals is a partition of $\mathbb{R}$.

By definition, a partition of $A$ is a pairwise disjoint collection of nonempty subsets of $A$ whose union is $A$. Recall from Section 2.3 that the definition of "pairwise disjoint" allows for the possibility that sets in a pairwise disjoint family may be equal.

Example. For the set $A = \{a, b, c, d, e\}$, the family $C = \{C_1, C_2, C_3\}$, where

$$C_1 = \{b, e\},\ C_2 = \{a, c, d\}, \text{ and } C_3 = \{b, e\},$$

is a partition of $A$ even though the sets $C_1$ and $C_3$ are not disjoint. The family $\{C_1, C_2, C_3\}$, is the same as the family $\{C_2, C_3\}$.

Let $W$ be a set of six people and $C = \{\text{blue, green, red, white}\}$. For each $c \in C$, let

$$B_c = \{x \in W : x \text{ is wearing clothing with color } c\}.$$

and let $\mathcal{B} = \{B_{\text{blue}}, B_{\text{green}}, B_{\text{red}}, B_{\text{white}}\}$. The family $\mathcal{B}$ may not be a partition of $W$ because any of the three parts of the definition might be violated. If no one is wearing red, then $B_{\text{red}}$ is empty, so condition (i) fails. If someone is wearing green only, while a second person is wearing green and blue, then the different sets $B_{\text{blue}}$ and $B_{\text{green}}$ overlap, in violation of condition (ii). If someone is wearing only yellow clothing, then that person does not belong to any set in $\mathcal{B}$, in violation of condition (iii).

The first half of the connection between partitions and equivalence relations is: Every equivalence relation on a set determines a partition of that set.

**Theorem 3.3.1** If $R$ is an equivalence relation on a nonempty set $A$, then $A/R$, the set of equivalence classes for $R$, is a partition of $A$.

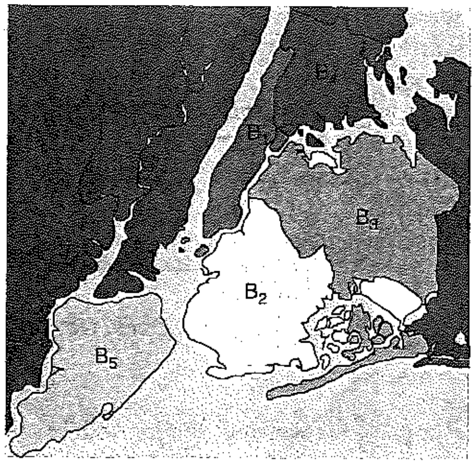
Proof. By Theorem 3.2.1 every equivalence class $x/R$ is a subset of $A$ and is nonempty because it contains $x$, and any two equivalence classes are either equal or disjoint. All that remains is to show that the union over $A/R$ is equal to $A$.

First $\bigcup_{x\in A} x/R \subseteq A$ because each $x/R \subseteq A$. To prove $A \subseteq \bigcup_{x\in A} x/R$, suppose $t \in A$. Since $t \in t/R$, $t \in \bigcup_{x\in A} x/R$. Thus $A = \bigcup_{x\in A} x/R$. ■

Example. Let $A = \{4, 5, 6, 7\}$ and $T$ be the equivalence relation

$$\{(4, 4), (5, 5), (6, 6), (7, 7), (5, 7), (7, 5), (7, 6), (6, 7), (5, 6), (6, 5)\}.$$

By Theorem 3.3.1, we can form a partition of $A$ by finding the equivalence classes of $T$. These are $4/T = \{4\}$ and $5/T = 6/T = 7/T = \{5, 6, 7\}$. The partition produced by $T$ is $A/R = \{\{4\}, \{5, 6, 7\}\}$.




The Five Boroughs of New York City
$B_1$: Manhattan
$B_2$: Brooklyn
$B_3$: Queens
$B_4$: The Bronx
$B_5$: Staten Island

**Figure 3.3.2**

New York City is divided into 5 boroughs (counties). The boroughs are labeled $B_1$ through $B_5$ in Figure 3.3.2. If $A$ is the set of all residents of New York City, then $A$ is partitioned into 5 subsets: the set of residents living in $B_1$, the residents living in $B_2$, and so on. How can we use this fact to define an equivalence relation on $A$? We say that two residents of New York City are equivalent iff they are in the same partition element; that is, they reside in the same borough.

The method we will use to produce an equivalence relation from a partition is based on this idea that two objects will be said to be related iff they belong to the same member of the partition. The next theorem proves that this method for defining a relation always produces an equivalence relation and, furthermore, the set of equivalence classes of the relation is the same as the original partition.

**Theorem 3.3.2** Let $\mathcal{P}$ be a partition of the nonempty set $A$. For $x$ and $y \in A$, define $x\ Q\ y$ iff there exists $C \in \mathcal{P}$ such that $x \in C$ and $y \in C$. Then

**(a)** $Q$ is an equivalence relation on $A$.
**(b)** $A/Q = \mathcal{P}$.

Proof.

**(a)** We prove $Q$ is transitive and leave the proofs of symmetry and reflexivity on $A$ for Exercise 10. Let $x, y, z \in A$. Assume $x\ Q\ y$ and $y\ Q\ z$. Then there are sets $C$ and $D$ in $\mathcal{P}$ such that $x, y \in C$ and $y, z \in D$. Since $\mathcal{P}$ is a partition of $A$, the sets $C$ and $D$ are either identical or disjoint; but since $y$ is an element of both sets, they cannot be disjoint. Hence, there is a set $C$ $(= D)$ that contains both $x$ and $z$, so that $x\ Q\ z$. Therefore, $Q$ is transitive.

**(b)** We first show $A/Q \subseteq \mathcal{P}$. Let $x/Q \in A/Q$. Then choose $B \in \mathcal{P}$ such that $x \in B$. We claim $x/Q = B$. If $y \in x/Q$, then $x\ Q\ y$. Then there is some $C \in \mathcal{P}$ such that $x \in C$ and $y \in C$. Since $x \in C \cap B$, $C = B$, so $y \in B$. On the other hand, if $y \in B$, then $x\ Q\ y$, and so $y \in x/Q$. Therefore, $x/Q = B$.

To show $\mathcal{P} \subseteq A/Q$, let $B \in \mathcal{P}$. As an element of a partition, $B \neq \varnothing$. Choose any $t \in B$; then we claim $B = t/Q$. If $s \in B$, then $t \, Q \, s$, so $s \in t/Q$. On the other hand, if $s \in t/Q$, then $t \, Q \, s$; so $s$ and $t$ are elements of the same member of $\mathcal{P}$, which must be $B$. ■

**Example.** Let $A = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\{1\}, \{2, 3\}, \{4\}\}$ be a partition of $A$ with three sets. The equivalence relation $Q$ associated with $\mathcal{P}$ is $\{(1, 1), (2, 2), (3, 3), (4, 4), (2, 3), (3, 2)\}$. The three equivalence classes for $Q$ are $1/Q = \{1\}$, $2/Q = 3/Q = \{2, 3\}$, and $4/Q = \{4\}$. The set of all equivalence classes is precisely $\mathcal{P}$.

**Example.** The set $\mathcal{A} = \{A_0, A_1, A_2, A_3\}$ is a partition of $\mathbb{Z}$, where

$$\begin{aligned} A_0 &= \{4k : k \in \mathbb{Z}\}. \\ A_1 &= \{4k + 1 : k \in \mathbb{Z}\}. \\ A_2 &= \{4k + 2 : k \in \mathbb{Z}\}. \\ A_3 &= \{4k + 3 : k \in \mathbb{Z}\}. \end{aligned}$$

Then integers $x$ and $y$ are in the same set $A_i$ iff $x = 4n + i$ and $y = 4m + i$ for some integers $n$ and $m$ or, in other words, iff $x - y$ is a multiple of 4. Thus, the equivalence relation associated with the partition $\mathcal{A}$ is the relation of congruence modulo 4 and each $A_i$ is the residue class of $i$ modulo 4, for $i = 0, 1, 2, 3$.

We have seen that every equivalence relation on a set determines a partition for the set and every partition of a set determines a corresponding equivalence relation on that set. Furthermore, if we start with an equivalence relation, the partition we make is the set of equivalence classes, and if we use that partition to form an equivalence relation, the relation formed is the relation we started with. Thus, each concept may be used to describe the other. This is to our advantage, for we may use partitions and equivalence relations interchangeably, choosing the one that lends itself more readily to the situation at hand.

## Exercises 3.3

1. Describe four different partitions of the set of all students enrolled at a university.
2. For the given set $A$, determine whether $\mathcal{P}$ is a partition of $A$.
   (a) $A = \{1, 2, 3, 4\}$, $\mathcal{P} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$
   (b) $A = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{P} = \{\{1, 2\}, \{3\}, \{4, 5\}\}$
   (c) $A = \{1, 2, 3, 4, 5, 6, 7\}$, $\mathcal{P} = \{\{1, 3\}, \{5, 6\}, \{2, 4\}, \{7\}\}$
   ★ (d) $A = \mathbb{N}$, $\mathcal{P} = \{1, 2, 3, 4, 5\} \cup \{n \in \mathbb{N} : n > 5\}$
   (e) $A = \mathbb{R}$, $\mathcal{P} = (-\infty, -1) \cup [-1, 1] \cup (1, \infty)$
   (f) $A = \mathbb{R}$, $\mathcal{P} = \{S_y : y \in \mathbb{R} \text{ and } y > 0\}$, where $S_y = \{x \in \mathbb{R} : x < y\}$

3. Describe the partition for each of the following equivalence relations.
   (a) For $x, y \in \mathbb{R}$, $x\, R\, y$ iff $x - y \in \mathbb{Z}$.
   ★ (b) For $n, m \in \mathbb{Z}$, $n\, R\, m$ iff $n$ and $m$ have the same tens digit.
   (c) For $x, y \in \mathbb{R}$, $x\, R\, y$ iff $\sin x = \sin y$.
   (d) For $x, y \in \mathbb{R}$, $x\, R\, y$ iff $x^2 = y^2$.
   (e) For $(x, y)$ and $(u, v) \in \mathbb{R} \times \mathbb{R}$, $(x, y)\, S\, (u, v)$ iff $xy = uv = 0$ or $xyuv > 0$.
   (f) $(x, y)\, R\, (u, v)$ iff $x + v = y + u$.

4. Let $C = \{i, -1, -i, 1\}$, where $i^2 = -1$. The relation $R$ on $C$ given by $x\, R\, y$ iff $xy = \pm 1$ is an equivalence relation on $C$. Give the partition of $C$ associated with $R$.

5. Let $C$ be as in Exercise 4. The relation $S$ on $C \times C$ given by $(x, y)\, S\, (u, v)$ iff $xy = uv$ is an equivalence relation. Give the partition of $C \times C$ associated with $S$.

6. Describe the equivalence relation on each of the following sets with the given partition.
   (a) $\mathbb{N}$, $\{\{1, 2, \ldots 9\}, \{10, 11, \ldots 99\}, \{100, 101, \ldots 999\}, \ldots\}$
   ★ (b) $\mathbb{Z}$, $\{\ldots, \{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}, \{3, 4, 5, \ldots\}\}$
   (c) $\mathbb{R}$, $\{(-\infty, 0), \{0\}, (0, \infty)\}$
   ★ (d) $\mathbb{R}$, $\{\ldots, (-3, -2), \{-2\}, (-2, -1), \{-1\}, (-1, 0), \{0\}, (0, 1), \{1\}, (1, 2), \{2\}, (2, 3), \ldots\}$
   (e) $\mathbb{Z}$, $\{A, B\}$, where $A = \{x \in \mathbb{Z}: x < 3\}$ and $B = \mathbb{Z} - A$

7. For each $a \in \mathbb{R}$, let $A_a = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = a - x^2\}$.
   (a) Sketch a graph of the set $A_a$ for $a = -2, -1, 0, 1,$ and 2.
   (b) Prove that $\{A_a: a \in \mathbb{R}\}$ is a partition of $\mathbb{R} \times \mathbb{R}$.
   (c) Describe the equivalence relation associated with this partition.

8. List the ordered pairs in the equivalence relation on $A = \{1, 2, 3, 4, 5\}$ associated with these partitions:
   ★ (a) $\{\{1, 2\}, \{3, 4, 5\}\}$ (b) $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$
   (c) $\{\{2, 3, 4, 5\}, \{1\}\}$

9. Partition the set $D = \{1, 2, 3, 4, 5, 6, 7\}$ into two subsets: those symbols made from straight line segments only (like 4), and those that are drawn with at least one curved segment (like 2). Describe or draw the digraph of the corresponding equivalence relation on $D$.

10. Complete the proof of Theorem 3.3.2 by proving that if $\mathcal{P}$ is a partition of $A$, and $x\, Q\, y$ iff there exists $C \in \mathcal{P}$ such that $x \in C$ and $y \in C$, then
    (a) $Q$ is symmetric.
    (b) $Q$ is reflexive on $A$.

★ 11. Let $R$ be a relation on a set $A$ that is reflexive and symmetric but not transitive. Let $R(x) = \{y: x\, R\, y\}$. [Note that $R(x)$ is the same as $x/R$ except that $R$ is not an equivalence relation in this exercise.] Does the set $\mathcal{A} = \{R(x): x \in A\}$ always form a partition of $A$? Prove that your answer is correct.

12. Repeat Exercise 11, assuming $R$ is reflexive and transitive but not symmetric.
13. Repeat Exercise 11, assuming $R$ is symmetric and transitive but not reflexive.
14. Let $A$ be a set with at least three elements.
   ★ (a) If $\mathcal{P} = \{B_1, B_2\}$ is a partition of $A$ with $B_1 \neq B_2$, is $\{B_1^c, B_2^c\}$ a partition of $A$? Explain. What if $B_1 = B_2$?
   (b) If $\mathcal{P} = \{B_1, B_2, B_3\}$ is a partition of $A$, is $\{B_1^c, B_2^c, B_3^c\}$ a partition of $A$? Explain. Consider the possibility that two or more of the elements of $\mathcal{P}$ may be equal.
   (c) If $\mathcal{P} = \{B_1, B_2\}$ is a partition of $A$, $\mathcal{C}_1$ is a partition of $B_1$, and $\mathcal{C}_2$ is a partition of $B_2$, and $B_1 \neq B_2$, prove that $\mathcal{C}_1 \cup \mathcal{C}_2$ is a partition of $A$.

*Proofs to Grade*

15. Assign a grade of A (correct), C (partially correct), or F (failure) to each. Justify assignments of grades other than A.
   (a) **Claim.** Let $R$ be an equivalence relation on the set $A$, and let $x$, $y$, and $z$ be elements of $A$. If $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$.
   ***"Proof."*** Assume that $x \in y/R$ and $z \in x/R$. Then $y \, R \, x$ and $x \, R \, z$. By transitivity, $y \, R \, z$, so $z \in y/R$. Therefore, if $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$. ■
   (b) **Claim.** Let $R$ be an equivalence relation on the set $A$, and let $x$, $y$, and $z$ be elements of $A$. If $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$.
   ***"Proof."*** Assume that $x \in y/R$ and assume that $z \in y/R$. Then $y \, R \, x$ and $y \, R \, z$. By symmetry, $x \, R \, y$, and by transitivity, $x \, R \, z$. Therefore, $z \in x/R$. We conclude that if $x \in y/R$ and $z \notin x/R$, then $z \notin y/R$. ■
   (c) **Claim.** If $\mathcal{A}$ is a partition of a set $A$ and $\mathcal{B}$ is a partition of a set $B$, then $\mathcal{A} \cup \mathcal{B}$ is a partition of $A \cup B$.
   ***"Proof."***
      (i) If $X \in \mathcal{A} \cup \mathcal{B}$, then $X \in \mathcal{A}$, or $X \in \mathcal{B}$. In either case $X \neq \varnothing$.
      (ii) If $X \in \mathcal{A} \cup \mathcal{B}$ and $Y \in \mathcal{A} \cup \mathcal{B}$, then $X \in \mathcal{A}$ and $Y \in \mathcal{A}$, or $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, or $X \in \mathcal{B}$ and $Y \in \mathcal{A}$, or $X \in \mathcal{B}$ and $Y \in \mathcal{B}$. Since both $\mathcal{A}$ and $\mathcal{B}$ are partitions, in each case either $X = Y$ or $X \cap Y = \varnothing$.
      (iii) Since $\bigcup_{X \in \mathcal{A}} X = A$ and $\bigcup_{x \in \mathcal{B}} X = B$, $\bigcup_{X \in \mathcal{A} \cup \mathcal{B}} X = A \cup B$. ■
   ★ (d) **Claim.** If $\mathcal{B}$ is a partition of $A$, and if $x \, Q \, y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$, then the relation $Q$ is symmetric.
   ***"Proof."*** First, $x \, Q \, y$ iff there exists $C \in \mathcal{B}$ such that $x \in C$ and $y \in C$. Also, $y \, Q \, x$ iff there exists $C \in \mathcal{B}$ such that $y \in C$ and $x \in C$. Therefore, $x \, Q \, y$ iff $y \, Q \, x$. ■

## 3.4 Ordering Relations

Familiar ordering relations for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ such as "less than," "greater than," and "less than or equal to" are basic to our understanding of number systems but they are not equivalence relations. For instance, $<$ is not reflexive on $\mathbb{R}$ because $3 < 3$ is false, and is not symmetric because $2 < \pi$ is true but $\pi < 2$ is false. The relation $<$ is transitive, because the conjunction $x < y$ and $y < z$ implies $x < z$. This section describes those properties of relations that characterize orderings like $<$ and $\leq$. We begin with some examples.

Example. In addition to transitivity and reflexivity on $\mathbb{R}$, the relation $\leq$ on $\mathbb{R}$ has two properties we have not previously considered. The first of these properties is **comparability**: every two elements of $\mathbb{R}$ are comparable. This means that for all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$. The other property is that for all $x, y \in \mathbb{R}$, if $x \leq y$ and $y \leq x$, then $x = y$.

Example. We saw earlier that the relation "divides" is reflexive on $\mathbb{N}$. While we did not use the term "transitive" in Section 1.4, in effect we proved in that section that "divides" is transitive. Two other properties of this relation are notable. If $a$ divides $b$ and $b$ divides $a$, then $a = b$. Also, there are elements of $\mathbb{N}$ that are not comparable. That is, there are natural numbers $x$ and $y$ (for example, 10 and 21) such that both "$x$ divides $y$" and "$y$ divides $x$" are false.

Example. Let $X$ be a set. The set inclusion relation $\subseteq$ on the power set of $X$ is reflexive on $\mathcal{P}(X)$ and transitive. Also, if $A$ and $B$ are subsets of $X$ with $A \subseteq B$ and $B \subseteq A$ then $A = B$. In this relation some pairs of elements are not comparable. For example, if $X = \{1, 2, 3, 4\}$, then $\{1, 3\}$ and $\{1, 4\}$ are elements of $\mathcal{P}(X)$ but both $\{1, 3\} \subseteq \{1, 4\}$ and $\{1, 4\} \subseteq \{1, 3\}$ are false.

Example. Let $Y$ be the relation "is the same age in years or younger than" on a fixed set $P$ of people. Then $Y$ is reflexive on $P$ and transitive. This relation also has the property that any two elements of $P$ are comparable. However, the relation $Y$ has a property that is undesirable for an ordering. If $a$ and $b$ are two different people in $P$, and both $a$ and $b$ are 20 years old, then $a \, Y \, b$ and $b \, Y \, a$, but $a \neq b$.

Although we find it acceptable in an ordering for two elements to not be comparable, we wish to avoid the situation in the previous example where two different objects are both related to each other. The property we want is called antisymmetry.

> DEFINITION A relation $R$ on a set $A$ is **antisymmetric** iff for all $x, y \in A$, if $x \, R \, y$ and $y \, R \, x$, then $x = y$.

Examples. We have already noted that the relations "divides" on $\mathbb{N}$, $\leq$ on $\mathbb{R}$, and $\subseteq$ on $\mathcal{P}(A)$ are antisymmetric. The relation $<$ differs from the relation $\leq$ on $\mathbb{R}$ because $<$ is not reflexive on $\mathbb{R}$. Like $\leq$, the relation $<$ is antisymmetric but for a different reason: the statement "For all $x$, $y$ in $\mathbb{R}$, if $x < y$ and $y < x$ then $x = y$" is true because the antecedent is false.

The relation "divides" is an antisymmetric relation on $\mathbb{N}$. However, "divides" is not an antisymmetric relation on $\mathbb{Z}$. For example, 6 divides $-6$ and $-6$ divides 6, but $6 \neq -6$.

Antisymmetry is an important concept for maintaining the chain of command in the military where the relation "can give orders to" must be explicit. It would be chaotic if two different officers could give orders to each other.

A relation may be antisymmetric and not symmetric, symmetric and not antisymmetric, both, or neither. See Exercise 2. In Exercise 3, you are asked to show that if $R$ is an antisymmetric relation, then $x\,R\,y$ and $x \neq y$ implies $y \not{R}\, x$. That is, the only possible symmetry that an antisymmetric relation may exhibit is that an object may be related to itself.

> DEFINITION A relation $R$ on a set $A$ is a **partial order** (or **partial ordering**) for $A$ if $R$ is reflexive on $A$, antisymmetric, and transitive. A set $A$ with partial order $R$ is called a **partially ordered set**, or **poset**.

Three relations discussed above: "divides" on $\mathbb{N}$, $\leq$ on $\mathbb{R}$, and $\subseteq$ on $\mathcal{P}(X)$ for any set $X$, are examples of partial orderings.

Example. Let $W$ be the relation on $\mathbb{N}$ given by $x\,W\,y$ iff $x + y$ is even and $x \leq y$. Then $W$ is a partial order. For example, $2\,W\,4$, $4\,W\,6$, $6\,W\,8, \ldots$, and $1\,W\,3$, $3\,W\,5$, $5\,W\,7, \ldots$, but we never have $m\,W\,n$ where $m$ and $n$ have opposite parity. We verify that $W$ is a partial order:

Proof.

**(i)** ⟨*Show W is reflexive on* $\mathbb{N}$.⟩ Let $x \in \mathbb{N}$. Then $x + x = 2x$ is even and $x \leq x$, so $x\,W\,x$.

**(ii)** ⟨*Show W is antisymmetric.*⟩ Suppose $x\,W\,y$ and $y\,W\,x$. Then $x + y$ is even, $x \leq y$, and $y \leq x$. By antisymmetry of $\leq$ on $\mathbb{N}$, $x = y$.

**(iii)** ⟨*Show W is transitive.*⟩ Suppose $x\,W\,y$ and $y\,W\,z$. Then $x \leq y$, $x + y$ is even, $y \leq z$, and $y + z$ is even. By transitivity of $\leq$ on $\mathbb{N}$, $x \leq z$. Also, $x + z$ is even because $x + z = (x + y) + (y + z) + (-2y)$ is the sum of three even numbers. Therefore, $x\,W\,z$. ∎

Suppose $R$ is a partial order on the set $A$ and $a$, $b$, $c$ are three distinct elements of $A$. Further suppose that $a\,R\,b$, $b\,R\,c$, and $c\,R\,a$. A portion of the digraph of $R$ is shown in Figure 3.4.1. The chain of relationships $a\,R\,b$, $b\,R\,c$, $c\,R\,a$ is called a closed path (of length 3) in the digraph. (See the next section for more about paths in graphs.) The path is closed because as we move from vertex to vertex along the path, we can start and end at the same vertex. From $a\,R\,b$ and $b\,R\,c$, by transitivity we must have $a\,R\,c$. (The arc from $a$ to $c$ is not shown in the portion of the digraph in Figure 3.4.1.) But $c\,R\,a$ is also true, and this contradicts the antisymmetry property of $R$. Using this reasoning, we conclude that the digraph of a partial order can never contain a closed path except for loops at individual vertices.

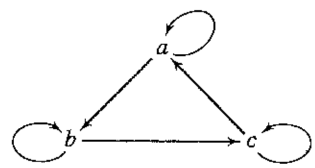



**Figure 3.4.1**

**Theorem 3.4.1** If $R$ is a partial order for a set $A$ and $x\,R\,x_1$, $x_1\,R\,x_2$, $x_2\,R\,x_3, \ldots, x_n\,R\,x$, then $x = x_1 = x_2 = x_3 = \cdots = x_n$.

Proof. ⟨*We prove this by induction on n.*⟩ For $n = 1$, suppose we have $x\,R\,x_1$ and $x_1\,R\,x$. By antisymmetry, we conclude that $x = x_1$.

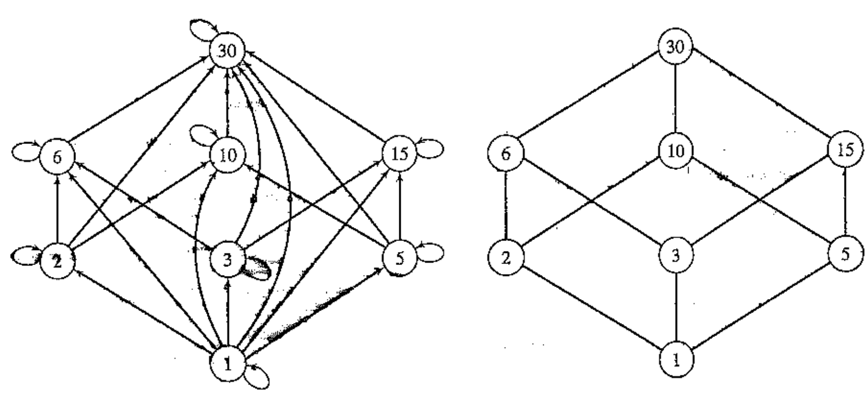
Now suppose that for some natural number $k$, whenever $x\,R\,x_1$, $x_1\,R\,x_2$, $x_2\,R\,x_3, \ldots, x_k\,R\,x$, then $x = x_1 = x_2 = x_3 = \cdots = x_k$ and suppose that $x\,R\,x_1$, $x_1\,R\,x_2$, $x_2\,R\,x_3, \ldots, x_k\,R\,x_{k+1}$, $x_{k+1}\,R\,x$. By transitivity ⟨*applied to* $x_k\,R\,x_{k+1}$ and $x_{k+1}\,R\,x$⟩ we have $x_k\,R\,x$. From $x\,R\,x_1$, $x_1\,R\,x_2, \ldots, x_k\,R\,x$ and the hypothesis of induction, we have $x = x_1 = x_2 = \cdots = x_k$. Since $x_k = x$ we have $x\,R\,x_{k+1}$ and $x_{k+1}\,R\,x$, so $x = x_{k+1}$. Therefore, $x = x_1 = x_2 = \cdots = x_{k+1}$. ■

DEFINITION Let $R$ be a partial ordering on a set $A$ and let $a, b \in A$ with $a \neq b$. Then $a$ is an **immediate predecessor** of $b$ iff $a\,R\,b$ and there does not exist $c \in A$ such that $a \neq c$, $b \neq c$, $a\,R\,c$ and $c\,R\,b$.

In other words, $a$ is an immediate predecessor of $b$ when $aRb$ and no other element lies "between" $a$ and $b$.

Example. For $A = \{1, 2, 3, 4, 5\}$, $\mathcal{P}(A)$ is partially ordered by the set inclusion relation $\subseteq$. For the set $\{2, 3, 5\}$, there are three immediate predecessors in $\mathcal{P}(A)$: $\{2, 3\}$, $\{2, 5\}$, and $\{3, 5\}$. The empty set has no immediate predecessor. Also, $\varnothing$ is the only immediate predecessor for $\{3\}$. We have $\{4\} \subseteq \{2, 4, 5\}$, but $\{4\}$ is not an immediate predecessor of $\{2, 4, 5\}$ because $\{4\} \neq \{4, 5\}$, $\{4, 5\} \neq \{2, 4, 5\}$, $\{4\} \subseteq \{4, 5\}$, and $\{4, 5\} \subseteq \{2, 4, 5\}$.

Let $M = \{1, 2, 3, 5, 6, 10, 15, 30\}$ be the set of all positive divisors of 30. The relation "divides" is a partial order for $M$ whose digraph is given in Figure 3.4.2(a). We can simplify the digraph significantly. First, since we know that every vertex must have a loop, we need not include them in the digraph. Also, since there are no closed paths, we can orient the digraph so that all edges point upward; thus we may eliminate the arrowheads, assuming that each edge has the arrowhead on the upper end. We can also remove edges that can be recovered by transitivity. For example, since there is an edge from 2 to 10 and another from 10 to 30, we do not need to include the edge from 2 to 30. In other words, we need only include those edges that relate immediate predecessors. The resulting simplified digraph, Figure 3.4.2(b), is called a **Hasse diagram** of the partial order "divides."

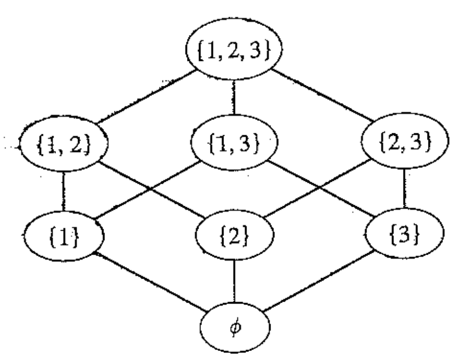



(a) Digraph of "divides."

(b) Hasse diagram for "divides."

**Figure 3.4.2**

Example. Let $A = \{1, 2, 3\}$. The Hasse diagram for $\mathcal{P}(A)$ partially ordered by $\subseteq$ is given in Figure 3.4.3. It bears a striking resemblance to Figure 3.4.2(b) for good reason. Except for the naming of the elements in the sets, the orderings are the same. In fact, it can be shown that every partial order is "the same" as the set inclusion relation on subsets of some set. Although we need the concepts of Chapter 4 to




Hasse diagram for $\subseteq$.

**Figure 3.4.3**

make precise what we mean by "same," Exercise 19 outlines how one might start to show this.

DEFINITIONS Let $R$ be a partial order for a set $A$. Let $B$ be any subset of $A$ and $a \in A$. Then

$a$ is an **upper bound** for $B$ iff $b \, R \, a$ for every $b \in B$.
$a$ is a **lower bound** for $B$ iff $a \, R \, b$ for every $b \in B$.
$a$ is a **least upper bound** for $B$ (or **supremum** for $B$) iff

(i) $a$ is an upper bound for $B$, and
(ii) $a \, R \, x$ for every upper bound $x$ for $B$.

$a$ is a **greatest lower bound** for $B$ (or **infimum** for $B$) iff

(i) $a$ is a lower bound for $B$, and
(ii) $x \, R \, a$ for every lower bound $x$ for $B$.

We write **sup (*B*)** to denote a supremum of $B$ and **inf (*B*)** for an infimum of $B$.

We shall soon see (Theorem 3.4.2) that there is at most one supremum and one infimum for a set.

Examples. For $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, let $B = \{\{1, 4, 5, 7\}, \{1, 4, 7, 8\}, \{2, 4, 7\}\}$. $B$ is a subset of $\mathcal{P}(A)$. Using the partial order $\subseteq$ for $\mathcal{P}(A)$, we see that $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is an upper bound for $B$ because

$$\begin{aligned} \{1, 4, 5, 7\} &\subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}, \\ \{1, 4, 7, 8\} &\subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}, \text{ and} \\ \{2, 4, 7\} &\subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}. \end{aligned}$$

Another upper bound for $B$ is $\{2, 4, 5, 7, 8, 9, 10\}$. The least upper bound for $B$ is $\sup(B) = \{1, 2, 4, 5, 7, 8\}$.

Elements of $\mathcal{P}(X)$ that are lower bounds for $B$ are $\varnothing$, $\{4\}$, $\{7\}$, and $\{4, 7\}$. The greatest lower bound for $B$ is $\inf(B) = \{4, 7\}$.

You should notice in the example above that sup(*B*) is the union of the sets in $B$ and inf(*B*) is the intersection of the sets in $B$. This is true in general: for any non-empty set $A$ with $\mathcal{P}(A)$ partially ordered by $\subseteq$, if $B$ is a set of subsets of $A$, then $\sup(B) = \bigcup_{X \in B} X$ and $\inf(B) = \bigcap_{X \in B} X$. See Exercise 14.

Example. Here are least upper bounds and greatest lower bounds for some subsets of $\mathbb{R}$ with the usual ordering $\leq$:

for $A = [0, 4)$, $\sup(A) = 4$ and $\inf(A) = 0$.
for $B = \{1, 6, 3, 9, 12, -4, 10\}$, $\sup(B) = 12$ and $\inf(B) = -4$.

for $C = \{2^k: k \in \mathbb{N}\}$, sup($C$) does not exist and inf($C$) = 2.
for $D = \{2^{-k}: k \in \mathbb{N}\}$, sup($D$) = $\frac{1}{2}$ and inf($D$) = 0.

**Example.** Let $A$ be the set of all positive divisors of 1000 with the ordering relation "divides" on $A$. Let $B = \{10, 20, 25, 100\}$. Both 500 and 1000 are upper bounds for $B$; the least upper bound is 100. The greatest lower bound for $B$ is 5. Note that for "divides," the least upper bound is the lcm (least common multiple) and the greatest lower bound is the gcd (greatest common divisor).

**Theorem 3.4.2** Let $R$ be a partial order for a set $A$ and $B \subseteq A$. Then if sup($B$) exists, it is unique. Also, if inf($B$) exists, it is unique.

**Proof.** Suppose that $x$ and $y$ are both least upper bounds for $B$. (*We prove that* $x = y$.) Since $x$ and $y$ are least upper bounds, then $x$ and $y$ are upper bounds. Since $x$ is an upper bound and $y$ is a least upper bound, we must have $y\, R\, x$. Likewise, since $y$ is an upper bound and $x$ is a least upper bound, we must have $x\, R\, y$. From $x\, R\, y$ and $y\, R\, x$, we conclude that $x = y$ by antisymmetry. Thus, if it exists, sup($B$) is unique.

The proof for inf($B$) is left as an exercise. ∎

We have seen examples of sets $B$ where, when they exist, the least upper and greatest lower bounds for $B$ are in $B$ and other examples where they are not in $B$.

> DEFINITION Let $R$ be a partial order for a set $A$. Let $B \subseteq A$. If the greatest lower bound for $B$ exists and is an element of $B$, it is called the **smallest element** (or **least element**) of $B$. If the least upper bound for $B$ is in $B$, it is called the **largest element** (or **greatest element**) of $B$.

The usual ordering of the number systems has the comparability property: for any $x$ and $y$, either $x \le y$ or $y \le x$. A partial ordering with this property is called linear.

> DEFINITION A partial ordering $R$ on $A$ is called a **linear order** (or **total order**) on $A$ if for any two elements $x$ and $y$ of $A$, either $x\, R\, y$ or $y\, R\, x$.

**Examples.** Each of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ with the ordering $\le$ is linearly ordered. $\mathcal{P}(A)$ with set inclusion, where $A = \{1, 2, 3\}$, is not a linearly ordered set because the two elements $\{1, 2\}$ and $\{1, 3\}$ cannot be compared. Likewise, the relation "divides" is not a linear order for $\mathbb{N}$ because 3 and 5 are not related (neither divides the other).

If $R$ is a linear order on $A$, then by antisymmetry, if $x$ and $y$ are distinct elements of $A$, $x\, R\, y$ or $y\, R\, x$ (but not both). The Hasse diagram for a linear ordered set is a set of points on a vertical line.

For a given linear order on a set it is not always true that every subset has a smallest or largest element. The set of integers with $\leq$ is linearly ordered but the set $B = \{1, 3, 5, 7, \ldots\}$ has neither upper bounds nor a least upper bound. Likewise, $\{-2, -4, -8, -16, -32, \ldots\}$ has no greatest lower bound (and hence no smallest element).

> DEFINITION Let $L$ be a linear ordering on a set $A$. $L$ is a **well ordering** on $A$ if every nonempty subset $B$ of $A$ contains a smallest element.

In Chapter 2 we proved the Well-Ordering Principle from the Principle of Mathematical Induction. Using the terminology of this section, the Well-Ordering Principle says that the natural numbers are well ordered by $\leq$. The integers, $\mathbb{Z}$, on the other hand, are not well ordered by $\leq$ because we have seen that $\{-2, -4, -8, -16, -32, \ldots\}$ is a nonempty subset that has no smallest element.

Finally, we state without proof a remarkable result.

Theorem 3.4.3 **Well-Ordering Theorem**

Every set can be well ordered.

The Well-Ordering Theorem should not be confused with the Well-Ordering Principle of Section 2.5, which is a property of the natural numbers. The theorem says for any nonempty set $A$ there is always a way to define a linear ordering on the set so that every nonempty subset of $A$ has a least element. Even the set of real numbers, which we know is not well ordered by the usual linear order $\leq$, has some other linear ordering so that $\mathbb{R}$ is well ordered by that ordering. The proof of the Well-Ordering Theorem requires a new property of sets, the Axiom of Choice. (See Section 5.5.)

## Exercises 3.4

1. Which of these relations on the given set are antisymmetric?
   - ★ (a) $A = \{1, 2, 3, 4, 5\}$, $R = \{(1, 3), (1, 1), (2, 4), (3, 2), (5, 4), (4, 2)\}$.
   - (b) $A = \{1, 2, 3, 4, 5\}$, $R = \{(1, 4), (1, 2), (2, 3), (3, 4), (5, 2), (4, 2), (1, 3)\}$.
   - ★ (c) $\mathbb{Z}$, $x\,R\,y$ iff $x^2 = y^2$.
   - (d) $\mathbb{R}$, $x\,R\,y$ iff $x \leq 2^y$.
   - (e) $\mathbb{R} \times \mathbb{R}$, $x\,S\,y$ iff $y = x - 1$.
   - ★ (f) $A = \{1, 2, 3, 4\}$, $R$ as given in the digraph: