## More Final Exam Review Problems

**Problem 1.** Let $f$ be a partial computable function. Explain why the function

$$g(n) = \begin{cases} 1 & \text{if } f(m)\downarrow \text{ for all } m \leq n \\ \uparrow & \text{otherwise} \end{cases}$$

is also partial computable.

**Solution.** To compute $g(n)$ we run the following loop:

Step 1. Set $m = 0$.

Step 2. Run $f(m)$. If $f(m)$ halts proceed to Step 3. If $f(m)$ never halts, we wait in this step forever (and so $g(n)\uparrow$).

Step 3. Check if $m = n$. If so, halt and output 1 (and so $g(n) = 1$). Otherwise increment $m$ by 1 and repeat Step 2.

**Problem 2.** Which sets $X$ satisfy $X \leq_m \emptyset$?

**Solution.** To have $X \leq_m \emptyset$, we need a computable function $f$ such that

$$n \in X \Leftrightarrow f(n) \in \emptyset$$

Since $f(n) \notin \emptyset$ (by the definition of $\emptyset$), we must have $n \notin X$ for all $n$. Therefore, the only set $X$ for which $X \leq_m \emptyset$ is $X = \emptyset$.

**Problem 3.** Which sets $X$ satisfy $X \leq_m \mathbb{N}$?

**Solution.** To have $X \leq_m \mathbb{N}$, we need a computable function $f$ such that

$$n \in X \Leftrightarrow f(n) \in \mathbb{N}$$

Since $f(n) \in \mathbb{N}$ no matter what $n$ is, we must have $n \in X$ for all $n$. Therefore, the only set $X$ for which $X \leq_m \mathbb{N}$ is $X = \mathbb{N}$.

**Problem 4.** Let $W$ be a c.e. set. Show that $W \leq_m K_0$, where $K_0 = \{\langle e, n\rangle \mid \varphi_e(n)\downarrow\}$.

**Solution.** Since $W$ is a c.e. set, we can fix an index $e$ such that $W = \text{domain}(\varphi_e)$. Let $f(n) = \langle e, n\rangle$. Since our pairing function is computable, $f$ is a computable function. We claim that $f$ witnesses that $W \leq_m K_0$.

$$\begin{aligned} n \in W &\Leftrightarrow \varphi_e(n)\downarrow \\ &\Leftrightarrow \langle e, n\rangle \in K_0 \\ &\Leftrightarrow f(n) \in K_0 \end{aligned}$$

**Problem 5.** Fix an index $e \in \mathbb{N}$ and let $A = \{i \mid \varphi_i = \varphi_e \text{ as partial functions}\}$.

- Prove that $A$ is an index set.
- Prove that $A \neq \emptyset$ and $A \neq \mathbb{N}$.
- Prove that $A$ is not computable.

**Solution.** To see $A$ is an index set, suppose $i \in A$ and $j \sim i$ (that is, $\varphi_j = \varphi_i$ as partial functions). We need to show that $j \in A$. Since $i \in A$, we know $\varphi_i = \varphi_e$ as partial functions, and so $\varphi_j = \varphi_e$ as partial functions. Therefore, $j \in A$ as required.

To see that $A \neq \emptyset$, note that $e \in A$ because $\varphi_e = \varphi_e$ are partial functions. To see $A \neq \mathbb{N}$, let $a$ be an index such that $\varphi_a(n) = 0$ for all $n$, and let $b$ be an index such that $\varphi_b(n) = 1$ for all $n$. Since $\varphi_e$ can be equal to at most one of $\varphi_a$ or $\varphi_b$, we have that either $a \notin A$ or $b \notin A$ (or both). Therefore, $A \neq \mathbb{N}$.

By Rice's theorem, $A$ is not computable because $A$ is an index set such that $A \neq \emptyset$ and $A \neq \mathbb{N}$.

**Problem 6.** Let $A = \{e \mid \text{domain}(\varphi_e) \leq_m K_0\}$.

- Prove that $A$ is an index set.
- Explain why $A$ is computable.
- Explain why these two facts do not contradict Rice's theorem.

**Solution.** To see $A$ is an index set, suppose $e \in A$ and $i \sim e$ (that is, $\varphi_i = \varphi_e$ as partial functions). We need to show that $i \in A$. Since $i \sim e$, we know that $\text{domain}(\varphi_i) = \text{domain}(\varphi_e)$. Since $e \in A$, we know $\text{domain}(\varphi_e) \leq_m K_0$. Therefore, $\text{domain}(\varphi_i) \leq_m K_0$ and so $i \in A$ by the definition of $A$.

To see why $A$ is computable, recall that by Problem 4, every c.e. set $W$ satisfies $W \leq_m K_0$. By definition, each set of the form $\text{domain}(\varphi_e)$ is c.e., and so $\text{domain}(\varphi_e) \leq_m K_0$ for every $e$. Therefore $A = \mathbb{N}$ and so $A$ is computable.

These facts do not contradiction Rice's theorem because Rice's theorem says that an index set is not computable when it is neither $\emptyset$ nor $\mathbb{N}$. The index set $A$ in this problem is equal to $\mathbb{N}$ and so Rice's theorem does not apply to it.