Final Exam Review Problems

These are review problems for the material on register machines and partial computable functions. About half the final exam will be on this material. The other half of the exam will be on classical and model propositional logic.

Problem 1. Show that the following functions are register machine computable: $n!$, n^m and $\max\{n, m\}$.

Recall that we used the notion of a *reduction* to show that several variations of the halting problem are not computable. For $A, B \subseteq \mathbb{N}$, we write $A \leq_m B$ if there is a (total) computable function f such that for all n, $n \in A \Leftrightarrow f(n) \in B$. The key point is that if A is not computable and $A \leq_m B$, then B is also not computable. The next few problems expand on this notion and use the following sets from class:

$$K = \{e : \varphi_e(e)\downarrow\}$$
$$K_0 = \{\langle e, n\rangle : \varphi_e(n)\downarrow\}$$
$$K_1 = \{e : \varphi_e(0)\downarrow\}$$

After proving that K is not computable (by a contradiction argument), we proved that $K \leq_m K_0$ and $K \leq_m K_1$ and so concluded that neither K_0 nor K_1 is computable.

Problem 2(a). Prove that $A \leq_m A$ for every set A (so the relation $\leq_m$ is reflexive).

2(b). Prove that if $A \leq_m B$ and $B \leq_m C$, then $A \leq_m C$ (so the relation $\leq_m$ is transitive).

2(c). Let $E = \{2n : n \in \mathbb{N}\}$ be the set of even numbers and $O = \{2n+1 : n \in \mathbb{N}\}$ be the set of odd numbers. Prove that $E \leq_m O$ and $O \leq_m E$. Explain why this example shows that $\leq_m$ is not an anti-symmetric relation.

Problem 3. Define a new relation $\equiv_m$ between sets as follows:

$$A \equiv_m B \Leftrightarrow A \leq_m B \text{ and } B \leq_m A.$$

Prove that $\equiv_m$ is an equivalence relation. That is, $\equiv_m$ is reflexive, symmetric and transitive.

Intuitively, you should think of $A \equiv_m B$ as saying that the sets A and B have equivalent computational content. Problem 4 below tells you that despite the fact that K_0 looks like the most complicated form of the halting problem, each of the variations are actually equally complicated.

Problem 4(a). Prove that $K_1 \leq_m K_0$ and that $K_0 \leq_m K$.

4(b). Using our results from class, show that $K \equiv_m K_0 \equiv_m K_1$.

Hint. To show $K_0 \leq_m K$, use the s-m-n theorem. As a reminder, it says:

s-m-n theorem: Let $f(x, y)$ be a partial computable function. There is a total computable function $s(x)$ such that for all x and y, $\varphi_{s(x)}(y) = f(x, y)$ as partial functions.

Try applying the s-m-n theorem to the following partial computable function:

$$f(\langle x_0, x_1 \rangle, y) = \begin{cases} 1 & \text{if } \varphi_{x_0}(x_1)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

This function is really a function of two variables $f(x, y)$, but we interpret the variable x as the code for a pair $\langle x_0, x_1 \rangle$ and so write the function as $f(\langle x_0, x_1 \rangle, y)$.

Problem 5. Consider the set $\text{TOT} = \{e : \varphi_e \text{ is total}\} = \{e : \varphi_e(n)\downarrow \text{ for all } n\}$. Prove that TOT is not computable.

Hint. There are a lot of ways to do Problem 5 and it is worth trying to do it in several different ways. One method is to prove it by contradiction as we did when we proved K is not computable. A second method is to show $K \leq_m \text{TOT}$ using the s-m-n theorem. A third method is to use Rice's theorem to help you. (See Problem 7 below for another application of Rice's theorem.)

For the next problem, you need the following definition. Let $g(x)$ be a partial function. A (total) function $f : \mathbb{N} \to \mathbb{N}$ is called an *extension of* g if $f(n) = g(n)$ for every $n \in \text{domain}(g)$. That is, the function $f(x)$ fills in all of the places where $g(x)$ is undefined (and there is no restriction on what values $f(x)$ uses to fill in these gaps).

Problem 6. Consider the following partial computable function.

$$g(e) = \begin{cases} \varphi_e(e) + 1 & \text{if } \varphi_e(e)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

Prove that there is no (total) computable function f such that f is an extension of g.

Hint. Try to do this by contradiction. If you are stuck, look back at our proof that K is not computable.

For the next problem, recall that we say e and i *are equivalent indices* (and write $e \sim i$) if $\varphi_e = \varphi_i$ as partial functions. An *index set* is a set $A \subseteq \mathbb{N}$ such that for all e and i, if $e \in A$ and $e \sim i$, then $i \in A$.

Rice's Theorem. Let $A \subseteq \mathbb{N}$ be an index set. If $A \neq \emptyset$ and $A \neq \mathbb{N}$, then A is not computable.

Problem 7(a). Consider the set Ext $= \{e : \varphi_e$ has a total computable extension$\}$. Prove that EXT is an index set.

7(b). Prove that Ext is not computable.

Hint. For 7(b), use Rice's Theorem. Be sure to explain why Ext is neither empty nor equal to $\mathbb{N}$.

Problem 8(a). Consider the set Fin $= \{e : \varphi_e$ has a finite domain$\}$. Prove that Fin is an index set.

8(b). Prove that Fin is not computable.

Problem 9. Let A be a nonempty c.e. set. Prove that there is a total computable function g such that $A = \text{range}(g)$.

Problem 10. Let A be an infinite c.e. set. Prove that there is a total computable function g such that g is one-to-one and $A = \text{range}(g)$.