Exam 2 Review Problems

Problem 1. Prove that if $M = (\mathcal{F}, R, v)$ is a symmetric model (i.e. R is a symmetric binary relation), then $M \models \alpha \to \Box\Diamond\alpha$ for every modal proposition α.

Comment. The axiom scheme $\alpha \to \Box\Diamond\alpha$ is called Axiom B in the modal logic literature. It is characterized by the class of symmetric models in the same sense as the other axioms we have looked at in class.

Problem 2. A binary relation is called *weakly dense* if whenever xRy holds, there is a z such that xRz and zRy hold. Intuitively, you can think of z as "sitting between" x and y. However, note that z does not have to be different from x and y – this is why the notion is called *weakly* dense.

2(a). Prove that if R is reflexive, then R is weakly dense.

2(b). Give an example of a weakly dense relation R which is not reflexive.

2(c). Prove that if $M = (\mathcal{F}, R, v)$ is a weakly dense model (i.e. R is a weakly dense binary relation), then $M \models \Box\Box\alpha \to \Box\alpha$ for every modal proposition α.

2(d). Give a model $M = (\mathcal{F}, R, v)$ such that $M \models \Box\Box A \to \Box A$ but $M \not\models \Box A \to A$.

Problem 3. The following exercises are examples of a general notion of duality in modal logic. Let α be an arbitrary modal proposition.

3(a). Show that if M is reflexive, then $M \models \alpha \to \Diamond\alpha$. Then show that $T \vdash \alpha \to \Diamond\alpha$ by giving a tableau proof of $\alpha \to \Diamond\alpha$ using the reflexive development rule.

3(b). Show that if M is euclidean, then $M \models \Diamond\Box\alpha \to \Box\alpha$. Then show that $5 \vdash \Diamond\Box\alpha \to \Box\alpha$ by giving a tableau proof of $\Diamond\Box\alpha \to \Box\alpha$ using the euclidean development rule.

Problem 4. Let α be an arbitrary modal formula.

4(a). Show that if M is symmetric, then $M \models \Diamond\Box\alpha \to \alpha$.

4(b). By the comment after Problem 1, Axiom B is the scheme $\alpha \to \Box\Diamond\alpha$ and the collection of symmetric models is sound and complete for Axiom B. Write down a symmetric tableau development rule that captures the notion of symmetric models. Show that $B \vdash \Diamond\Box\alpha \to \alpha$ by giving a tableau proof using your symmetric development rule.

Problem 5. Recall that S5 is the modal system which includes Axiom 5 and Axiom T. It is characterized by models which are reflexive and euclidean. Show that $\mathrm{S5} \vdash \alpha \to \Box\Diamond\alpha$ for every modal proposition α.

Hint. Show that $\alpha \to \Box\Diamond\alpha$ is tableau provable using the reflexive and euclidean development rules.