# Math 3265 Exam 1 Review Terms

On the exam, I am likely to ask some multiple part questions which begin with asking you to define a mathematical term we have learned, or state a theorem we have learned. The next part (or parts) of the question will ask you to prove something that almost certainly uses that term or theorem.

Here are the terms and theorems you should be ready to state precisely on the exam. Let $\Sigma$ be a set of propositions and let $\alpha$ be a proposition.

- You should be able to state the following properties of a binary relation on a set: reflexive, irreflexive, symmetric, anti-symmetric, transitive, trichotomy.
- You should be able to state the properties required for a binary relation to be a (strict or not) partial order, (strict or not) linear order, or an equivalence relation.
- **Model of $\boldsymbol{\Sigma}$**: A valuation $\mathcal{V}$ is a model of $\Sigma$ if and only if $\mathcal{V}(\sigma) = T$ for all $\sigma \in \Sigma$.
- **Tautology**: A proposition $\alpha$ is a tautology if and only if $\mathcal{V}(\alpha) = T$ for all valuations $\mathcal{V}$.
- $\models \alpha$: For all valuations $\mathcal{V}$, $\mathcal{V}(\alpha) = T$. (This is the same as saying $\alpha$ is a tautology.)
- $\alpha$ **is a consequence of** $\Sigma$: For every model $\mathcal{V}$ of $\Sigma$, $\mathcal{V}(\alpha) = T$.
- $\boldsymbol{\Sigma} \models \alpha$: For every model $\mathcal{V}$ of $\Sigma$, $\mathcal{V}(\alpha) = T$. (This is the same as saying $\alpha$ is a consequence of $\Sigma$.)
- $\alpha$ **is (tableau) provable**: There is a contradictory tableau with root $F\ \alpha$.
- $\vdash \alpha$: There is a contradictory tableau with root $F\ \alpha$. (This is the same as saying $\alpha$ is (tableau) provable.)
- $\alpha$ **is (tableau) provable from** $\Sigma$: There is a contradictory tableau from $\Sigma$ with root $F\ \alpha$.
- $\boldsymbol{\Sigma} \vdash \alpha$: There is a contradictory tableau from $\Sigma$ with root $F\ \alpha$. (This is the same as saying $\alpha$ is (tableau) provable from $\Sigma$.)
- **Soundness Theorem**: $\Sigma \vdash \alpha$ implies $\Sigma \models \alpha$.
- **Completeness Theorem**: $\Sigma \models \alpha$ implies $\Sigma \vdash \alpha$.

I also wanted to make a couple of comments on Homework 3 to point out some mistakes that showed up on several papers.

- The following mistakes occurred on a number of tableau proofs. The thing to remember is that tableau proofs are formal proofs – they are not proofs in the usual mathematical sense. So, you have to follow the rules for generating a tableau exactly (except for repeating the root of each atomic tableau).
  - When decomposing a proposition using an atomic tableau, you need to put down the full atomic tableau (i.e. all the entries along each path in the atomic tableau) even if you only need some of these entries to get a contradiction along a path.
  - In order for a path in a tableau to be contradictory, it has to contain $T\ \alpha$ and $F\ \alpha$ for some proposition $\alpha$. It is not good enough to see that a contradiction is about to arise. For example, having $T\ \alpha$ and $T\ \neg\alpha$ along a path is not enough to make the path contradictory. You need to attach the atomic tableau for $T\ \neg\alpha$ to get $F\ \alpha$ along the path before it becomes contradictory.
- In problems like Problem 1 and Problem 2, it is important to make the structure of your argument clear. Here is some advice for tackling mathematical proofs.
  - Make sure you are clear about exactly what each terms means.
  - Write down very clearly what your assumptions are and what you are trying to prove. In simple proofs, just getting these points straight will show you how to complete the proof.
  - If you are proving an "if and only if" statement, make sure you prove both implications and make sure you are clear during the proof which direction you are working on. Sometimes you can prove both directions together, but it is often helpful to consider the directions separately.

Let me illustrate some of the ideas in the second bullet point by giving a couple of correct proofs for Problem 2 that look different from the one I wrote in the solutions.

**Solution 1 to Problem 2.** We consider the two directions separately.

First, we prove that if $\Sigma \models \alpha$, then $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology. For this direction, we assume that $\Sigma \models \alpha$, which means that every valuation $\mathcal{V}$ that models $\Sigma$ satisfies $\mathcal{V}(\alpha) = T$. We need to show that $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology, which means that $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = T$ for every valuation $\mathcal{V}$.

So, let $\mathcal{V}$ be a valuation. We need to show that $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = T$. We break into two cases.

- **Case 1.** Suppose that $\mathcal{V}(\sigma_i) = F$ for at least one $i$. In this case, $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = F$, and so $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = T$.

- **Case 2.** The remaining case is when $\mathcal{V}(\sigma_i) = T$ for all $i$. In this case, $\mathcal{V}$ is a model of $\Sigma$. By our assumption that $\Sigma \models \alpha$, it follows that $\mathcal{V}(\alpha) = T$. Therefore, we have $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = T$ and $\mathcal{V}(\alpha) = T$, so $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = T$.

These two case compete the proof that $\Sigma \vdash \alpha$ implies that $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology.

Second, we prove that if $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology, then $\Sigma \models \alpha$. We assume that $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology, and we need to show that $\Sigma \models \alpha$. That is, we need to show that if a valuation $\mathcal{V}$ is a model of $\Sigma$, then $\mathcal{V}(\alpha) = T$.

So, fix a valuation $\mathcal{V}$ which is a model of $\Sigma$. This means that $\mathcal{V}(\sigma_i) = T$ for all $i$, and hence that $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = T$. Since $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology, we know $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = T$, and so we must have $\mathcal{V}(\alpha) = T$, which is what we needed to show to finish this direction of the proof.

**Solution 2 to Problem 2.** Instead of tackling the problem head-on, we try to prove the equivalent statement: $\Sigma \not\models \alpha$ if and only if $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology. In fancy terms, we consider the contrapositive of each direction.

First, we prove that if $\Sigma \not\models \alpha$, then $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology. For this direction, we assume that $\Sigma \not\models \alpha$, which means we know there is a valuation $\mathcal{V}$ which models $\Sigma$ but has $\mathcal{V}(\alpha) = F$. We need to show that $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology, which means that we need to show there is a valuation $\mathcal{V}$ for which $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = F$.

So, let $\mathcal{V}$ be a valuation which models $\Sigma$ but has $\mathcal{V}(\alpha) = F$ (from the assumption that $\Sigma \not\models \alpha$). We show that $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = F$. Since $\mathcal{V}$ models $\Sigma$, we have $\mathcal{V}(\sigma_i) = T$ for all $i$ and hence $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = T$. Since $\mathcal{V}(\alpha) = F$, this means that $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = F$, completing this direction.

Second, we prove that if $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology, then $\Sigma \not\models \alpha$. For this direction, we assume that $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology and we show that there is a model $\mathcal{V}$ of $\Sigma$ for which $\mathcal{V}(\alpha) = F$.

Since $(\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$ is not a tautology, there is a valuation $\mathcal{V}$ such that $\mathcal{V}((\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha) = F$. By the truth table for $\to$, we must have $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = T$ and $\mathcal{V}(\alpha) = F$. From $\mathcal{V}(\sigma_1 \wedge \cdots \wedge \sigma_n) = T$, it follows that $\mathcal{V}(\sigma_i) = T$ for all $i$, and hence $\mathcal{V}$ is a model of $\Sigma$. Therefore, $\mathcal{V}$ is a model of $\Sigma$ for which $\mathcal{V}(\alpha) = F$, completing this direction.

Notice that in each of these solutions to Problem 2, the structure of the argument is emphasized clearly. At each point, it is laid out what we are assuming and what we need to show. The assumptions and the desired conclusions start off as higher level statements such as $\Sigma \models \alpha$, so the first thing we do is to simplify the statements by writing down what they mean. Once that is done, there is relatively little left that is necessary to connect the assumptions to the desired conclusions.