Math 3265 Homework 5: Due Monday March 26

Problem 1. Let α and β be arbitrary modal propositions. Give tableau proofs for the following modal propositions.

1(a). $\neg\Diamond\neg\alpha \to \Box\alpha$

1(b). $\Box(\alpha \vee \neg\beta) \to (\Diamond\beta \to \Diamond\alpha)$

1(c). $(\Box(\alpha \to \beta) \wedge \Box\alpha) \to \Box\beta$

1(d). $(\neg\Diamond\alpha \wedge \Diamond\beta) \to \Diamond(\neg\alpha \wedge \beta)$

Problem 2. Let α, β and γ be arbitrary modal propositions. Give tableau proofs for the following deductions from premises.

2(a). $\neg\alpha \vdash \neg\Diamond\alpha$

2(b). $(\alpha \wedge \beta) \to \gamma \vdash (\Box\alpha \wedge \Box\beta) \to \Box\gamma$

2(c). $\alpha \to \beta \vdash \Diamond\alpha \to \Diamond\beta$

Problem 3. A binary relation R is called *serial* if for every x, there is a y such that xRy holds. A model $M = (\mathcal{F}, R, v)$ is called *serial* if the accessibility relation R is serial. Let M be a serial model. Prove that for any modal proposition α, $M \vDash \Box\alpha \to \Diamond\alpha$.

Hint. Fix a serial model $M = (\mathcal{F}, R, v)$. You need to show that for all worlds $w \in \mathcal{F}$, $w \Vdash_M \Box\alpha \to \Diamond\alpha$. So, assume that $w \Vdash_M \Box\alpha$ and show that $w \Vdash_M \Diamond\alpha$.

Problem 4. A binary relation is called *euclidean* if whenever xRy and xRz hold, then yRz holds. (Note that this property implies that zRy holds as well.) A model $M = (\mathcal{F}, R, v)$ is called *eucliean* if the accessibility relation R is euclidean. Let M be a euclidean model. Prove that for any modal proposition α,

$$M \models \Diamond\alpha \to \Box\Diamond\alpha \qquad \text{and} \qquad M \models \neg\Box\alpha \to \Box\neg\Box\alpha$$

Hint. Fix a euclidean model $M = (\mathcal{F}, R, v)$. First, you need to show that for all worlds $w \in \mathcal{F}$, $w \Vdash_M \Diamond\alpha \to \Box\Diamond\alpha$. So, assume $w \Vdash_M \Diamond\alpha$ and show $w \Vdash_M \Box\Diamond\alpha$. Follow the same pattern for the second proposition.

Review Problems

You do not need to hand in any of these problems. They are just review problems to help you study for the exam.

Review Problem 1. For each of the following modal propositions, either give a modal proof or give a counter-model showing the proposition is not valid.

- $\Box(A \to B) \to (\Diamond A \to \Diamond B)$
- $A \to \Box\Diamond A$
- $\Box\Diamond A \to A$
- $\Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$
- $\Box(A \wedge \Box B) \to (\Box A \vee \Box B)$
- $(\Box A \wedge \Diamond B) \to \Diamond(A \wedge B)$
- $(\Diamond A \to \Box B) \to \Box(A \to B)$
- $(\Diamond A \wedge \Box B) \to \Diamond(A \wedge \Box B)$

Review Problem 2. Give tableau proofs for the following deductions from premises.

- $\alpha \to \beta \vdash \Box\alpha \to \Box\beta$
- $\alpha \to (\beta \vee \gamma) \vdash \Diamond\alpha \to (\Diamond\beta \vee \Diamond\gamma)$