Math 3265 Homework 3: Due Friday February 23

Let Σ be a (possibly infinite) set of propositions. Recall that a valuation $\mathcal{V}$ is a model of Σ if $\mathcal{V}(\sigma) = T$ for all $\sigma \in \Sigma$. Recall that we write $\Sigma \models \alpha$, and say α is a consequence of Σ, if for every valuation $\mathcal{V}$ which is a model of Σ, $\mathcal{V}$ is also a model of α (i.e. $\mathcal{V}(\alpha) = T$). Let

$$\mathrm{Cn}(\Sigma) = \{\alpha : \Sigma \models \alpha\}$$
$$\mathrm{Mod}(\Sigma) = \{\mathcal{V} : \mathcal{V} \text{ is a model of } \Sigma\}$$

That is, $\mathrm{Cn}(\Sigma)$ is the set of consequences of Σ and $\mathrm{Mod}(\Sigma)$ is the set of valuations which are models of Σ.

Problem 1. Let Σ_1 and Σ_2 be sets of propositions such that $\Sigma_1 \subseteq \Sigma_2$. Explain why $\mathrm{Cn}(\Sigma_1) \subseteq \mathrm{Cn}(\Sigma_2)$, and $\mathrm{Mod}(\Sigma_2) \subseteq \mathrm{Mod}(\Sigma_1)$.

Problem 2. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a finite set of propositions. Prove that $\Sigma \models \alpha$ if and only if $(\sigma_1 \wedge \sigma_2 \wedge \cdots \wedge \sigma_n) \to \alpha$ is a tautology.

Problem 3. Give tableau proofs for the following propositions for any α, β and γ.

3(a). $(\alpha \to \beta) \leftrightarrow (\neg\beta \to \neg\alpha)$

3(b). $(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$

3(c). $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

3(d). $(\alpha \to (\beta \to \gamma)) \leftrightarrow ((\alpha \wedge \beta) \to \gamma)$

Problem 4. Give tableau proofs for each of the following deductions from premises.

4(a). $\{(A \vee B) \to (A \wedge B), A \to \neg B\} \vdash \neg A$.

4(b). $\{C \to (A \vee B), A \to \neg C, B \to \neg C\} \vdash \neg C$.

4(c). $\{\neg\neg A, C \to (\neg A \vee B)\} \vdash A \wedge (C \to B)$.

Problem 5. Does $\{A \vee B, A \to \neg B, A \to (B \vee C)\} \vdash C$? If so, give a tableau proof. If not, give a valuation $\mathcal{V}$ such that $\mathcal{V}(A \vee B) = \mathcal{V}(A \to \neg B) = \mathcal{V}(A \to (B \vee C)) = T$ and $\mathcal{V}(C) = F$.

Problem 6. Let $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$ be a finite set of propositions. Prove that the following notions are all equivalent. (You can use Problem 2 to help you.)

(A). $\Sigma \models \alpha$.

(B). $\models (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$.

(C). $\Sigma \vdash \alpha$.

(D). $\vdash (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \alpha$.

Practice Problems

You do not need to hand in the following problems. They are extra practice problems similar to the ones on this homework to help you prepare for the exams.

Practice Problems 1. Do Problems 1-9 in the Exercises in Section 4 of the notes on classical propositional logic.

Practice Problems 2. Prove all the parts of Proposition 3.10 in Section 3 of the notes on classical propositional logic.