Math 3265 Homework 4: Due Friday March 9

Problem 1. Let α and β be arbitrary model propositions. Prove that the following model propositions are valid.

1(a). $\neg\Diamond\neg\alpha \rightarrow \Box\alpha$

1(b). $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

Hint. Recall that for a model proposition to be valid, it must be forced at every world in every model. So, fix an arbitrary model $M = (\mathcal{F}, R, v)$ and an arbitrary world $w \in \mathcal{F}$. Show that w forces each of the statements above.

Problem 2. Let A and B be propositional letters. Prove that each of the following modal propositions is not valid.

2(a). $A \rightarrow \Diamond A$

2(b). $\Diamond A \rightarrow A$

2(c). $\Box A \rightarrow \Box\Box A$

2(d). $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$

Hint. For each of these modal propositions, you need to build a model $M = (\mathcal{F}, R, v)$ such that there is some world $w \in \mathcal{F}$ which does not force the modal proposition. When specifying your model, it is sufficient to only give the truth assignments for the propositional letters which occur in the modal proposition under consideration.

Problem 3. Consider the following modal frame $(\mathcal{F}, R)$: $\mathcal{F} = \{x, y, z\}$ and the relation holds only for xRy, xRz and yRz. Prove that for any modal proposition α, the modal proposition $\Box\alpha \rightarrow \Box\Box\alpha$ is forced in $(\mathcal{F}, R)$.

Hint. Start by fixing an arbitrary valuation function v and consider the model $M = (\mathcal{F}, R, v)$. You need to show that for each of the worlds x, y and z forces $\Box\alpha \rightarrow \Box\Box\alpha$.

Problem 4. Prove that for any modal proposition α,

$$\{\alpha \rightarrow \Box\alpha\} \models \Box\alpha \rightarrow \Box\Box\alpha$$

Hint. You need to show that for any model $M = (\mathcal{F}, R, v)$, if M forces $\alpha \rightarrow \Box\alpha$ (meaning every world $w \in \mathcal{F}$ forces $\alpha \rightarrow \Box\alpha$), M also forces $\Box\alpha \rightarrow \Box\Box\alpha$. So, fix an arbitrary model $M = (\mathcal{F}, R, v)$ and assume that for all worlds $w \in \mathcal{F}$, $w \Vdash_M \alpha \rightarrow \Box\alpha$. Prove that for all $w \in \mathcal{F}$, $w \Vdash_M \Box\alpha \rightarrow \Box\Box\alpha$.

Problem 5. Let A be a propositional letter. Prove that

$$\{\Box A \to A\} \not\models \Box A \to \Box\Box A$$

Hint. You need to build a model $M = (\mathcal{F}, R, v)$ such that for all worlds $w \in \mathcal{F}$, $w \Vdash_M \Box A \to A$, but there is at least one world $x \in \mathcal{F}$ such that $x \nVdash_M \Box A \to \Box\Box A$.

Problem 6. Let α be an arbitrary modal proposition. Prove that for any model $M = (\mathcal{F}, R, v)$ in which R is reflexive, $\Vdash_M \Box\alpha \to \alpha$.

Hint. Fix such a model M. You need to show that for each world $w \in \mathcal{F}$, $w \Vdash_M \Box\alpha \to \alpha$.

Problem 7. Let α be an arbitrary modal proposition. Prove that for any model $M = (\mathcal{F}, R, v)$ in which R is transitive, $\Vdash_M \Box\alpha \to \Box\Box\alpha$.