Review of vectors

We will work with real vectors, meaning vectors for which each component is a real number. With minimal changes, one can also work with complex vectors.

An **n-dimensional vector** is an n-tuple of real numbers, usually denoted by

$$\overline{v} = \langle v_1, v_2, \ldots, v_n \rangle$$

The real numbers $v_1, \ldots, v_n$ are called the **components** of $\overline{v}$. Often, we will use curved brackets rather than angled brackets. (One typically uses angled brackets to emphasize the difference between vectors and points.) We will use $\mathbb{R}^n$ or V_n to denote the set of all n-dimensional vectors. Since the exact (finite) dimension for the vectors is not important, we usually talk about a *vector* and leave the exact dimension implicit. For example, we will use $\overline{0}$ to denote the **zero vector**, namely the vector in which each component is 0. Note that there is actually a different zero vector for each dimension, but we assume that the dimension is fixed by the context.

A **scalar** is a real number. (Again, we could use complex numbers for scalars with minimal changes.) When working in linear algebra and multivariable calculus, it is important to keep in mind which quantities are *vectors* and which are *scalars*.

We introduced the following operations and concepts for vectors.

- Addition of vectors:

  $$\overline{v} + \overline{w} = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$$

  That is, we add vectors componentwise. (Note that the vectors have to have the same dimension and that the sum of two vectors is a vector.)

- Multiplication of a vector by a scalar:

  $$c\overline{v} = \langle cv_1, cv_2, \ldots, cv_n \rangle$$

  Note that a scalar times a vector is a vector.

- Dot product of two vectors:

  $$\overline{v} \cdot \overline{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

  Note that the dot product of two vectors is a scalar!

- The **norm** or **length** of $\overline{v}$ is

  $$\|\overline{v}\| = \sqrt{\overline{v} \cdot \overline{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

- The vectors $\overline{v}$ and $\overline{w}$ are **orthogonal** or **perpendicular** if $\overline{v} \cdot \overline{w} = 0$. They are **parallel** if $\overline{v} = c\overline{w}$ for a nonzero scalar c.

- For nonzero vectors $\overline{v}$ and $\overline{w}$, the **angle between** $\overline{v}$ and $\overline{w}$ is θ where

$$\cos(\theta) = \frac{\overline{v} \cdot \overline{w}}{\|\overline{v}\|\|\overline{w}\|}$$

- If $\overline{w}$ is nonzero, then the **projection of** $\overline{v}$ onto $\overline{w}$ is

$$\frac{\overline{v} \cdot \overline{w}}{\|\overline{w}\|^2}\overline{w}$$

You should look back over the basic properties of these operations – for example, the fact that $\|\overline{v}\| \geq 0$ for all vectors $\overline{v}$, and $\|\overline{v}\| = 0$ if and only if $\overline{v} = \overline{0}$. The two fundamental theorems we used many times in Chapter 12 are the **Cauchy–Schwarz Inequality** and the **Triangle Inequality**. You should remind yourself what these inequalities are.

The other important part of Chapter 12 to review are the notions of **(linear) span**, **(linear) independence** and **basis**. (I will drop the term "linear" since the only notions of span and independence we will use are the linear notions.) I will review each of these notions and give examples of calculations.

The main point to notice in all of the examples given is that questions about spans, independence and bases for $\mathbb{R}^n$ always reduce down to solving systems of linear equations. One of the goals of the first part of this course will be to use matrices to give a cleaner method for solving systems of linear equations, and hence answering these types of questions. However, before the matrix methods are useful, you need to be able to translate the questions about sets of vectors into questions about solutions of linear systems.

Review of linear combinations and spans

Let $S = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k\}$ be a set of vectors in $\mathbb{R}^n$. A vector $\overline{w}$ is a **linear combination of** $\overline{v}_1, \ldots, \overline{v}_k$ if there are scalars $c_1, \ldots, c_k$ such that

$$\overline{w} = \sum_{i=1}^{n} c_i \overline{v}_i = c_1\overline{v}_1 + c_2\overline{v}_2 + \cdots + c_k\overline{v}_k$$

If $\overline{w}$ is a linear combination of the vectors in S, then we say $\overline{w}$ is **spanned by** S. We define **the span of** S to be the set of all vectors spanned by S and we denote the span of S by $L(S)$.

Example 1. Let S be the following set of vectors in $\mathbb{R}^3$.

$$S = \{\langle 1, -2, 3\rangle, \langle 5, -13, -3\rangle\}$$

Is $\langle -3, 8, 1\rangle$ spanned by S? (Equivalently, is $\langle -3, 8, 1\rangle \in L(S)$? Also equivalently, is $\langle -3, 8, 1\rangle$ a linear combination of $\langle 1, -2, 3\rangle$ and $\langle 5, -13, -3\rangle$?)

Solution 1. The question is asking whether there are scalars (i.e. numbers) x and y such that

$$\langle -3, 8, 1\rangle = x\,\langle 1, -2, 3\rangle + y\,\langle 5, -13, -3\rangle\,?$$

(You will see in a second why I used x and y for my scalars instead of the c_1 and c_2 from the definition.) To figure out whether there are such scalars x and y, rewrite this condition a little bit.

$$\begin{aligned}\langle -3, 8, 1\rangle &= x\,\langle 1, -2, 3\rangle + y\,\langle 5, -13, -3\rangle \\ \Leftrightarrow \langle -3, 8, 1\rangle &= \langle x, -2x, 3x\rangle + \langle 5y, -13y, -3y\rangle \\ \Leftrightarrow \langle -3, 8, 1\rangle &= \langle x + 5y, -2x - 13y, 3x - 3y\rangle\end{aligned}$$

That is, we need to figure out if there is a solution to the set of (linear) equations

$$\begin{aligned}-3 &= x + 5y \\ 8 &= -2x - 13y \\ 1 &= 3x - 3y\end{aligned}$$

If there is a solution, then $\langle -3, 8, 1\rangle$ is spanned by S and each solution gives us a linear combination of $\langle 1, -2, 3\rangle$ and $\langle 5, -13, -3\rangle$ that yields $\langle -3, 8, 1\rangle$. (There could be more than one solution.) If there is no solution, then $\langle -3, 8, 1\rangle$ is not spanned by S.

So, we try to solve this system of equations. If we multiply the top equation by 2, the top two equations become

$$\begin{aligned}-6 &= 2x + 10y \\ 8 &= -2x - 13y\end{aligned}$$

Adding these equations gives $2 = -3y$, or in other words, $y = -2/3$. Plugging $y = -2/3$ into the top equation gives us

$$-3 = x - 10/3$$

or, in other words, $x = 1/3$. Therefore, the only possible solution to this system of equations is $x = 1/3$ and $y = -2/3$. If you plug these numbers into the second equation, you will see that they give a solution to the second equation. However, if you try to plug them into the third equation, they do not work.

Therefore, there is no solution to this system of equations and the vector $\langle -3, 8, 1\rangle$ is not spanned by S.

Example 2. Consider the set S of vectors in $\mathbb{R}^4$.

$$S = \{\langle 2, -1, 3, 6\rangle, \langle 4, 0, 1, 1\rangle, \langle 3, -2, 0, 0\rangle\}$$

Is $\langle 9, -8, 5, 11\rangle \in L(S)$? If so, write $\langle 9, -8, 5, 11\rangle$ as a linear combination of $\langle 2, -1, 3, 6\rangle$, $\langle 4, 0, 1, 1\rangle$ and $\langle 3, -2, 0, 0\rangle$.

Solution 2. This question is asking whether $\langle 9, -8, 5, 11\rangle$ can be written as a linear combination of $\langle 2, -1, 3, 6\rangle$, $\langle 4, 0, 1, 1\rangle$ and $\langle 3, -2, 0, 0\rangle$. Therefore, it is asking whether there are scalars x, y and z such that

$$\langle 9, -8, 5, 11\rangle = x\,\langle 2, -1, 3, 6\rangle + y\,\langle 4, 0, 1, 1\rangle + z\,\langle 3, -2, 0, 0\rangle\,?$$

Again, we start by rewriting this question to reduce it to a system of linear equations.

$$\begin{aligned} &\langle 9, -8, 5, 11\rangle = x\,\langle 2, -1, 3, 6\rangle + y\,\langle 4, 0, 1, 1\rangle + z\,\langle 3, -2, 0, 0\rangle \\ \Leftrightarrow\ &\langle 9, -8, 5, 11\rangle = \langle 2x, -x, 3x, 6x\rangle + \langle 4y, 0, y, y\rangle + \langle 3z, -2z, 0, 0\rangle \\ \Leftrightarrow\ &\langle 9, -8, 5, 11\rangle = \langle 2x + 4y + 3z, -x - 2z, 3x + y, 6x + y\rangle \end{aligned}$$

In other words, is there a solution to the system of equations

$$\begin{aligned} 9 &= 2x + 4y + 3z \\ -8 &= -x - 2z \\ 5 &= 3x + y \\ 11 &= 6x + y \end{aligned}$$

Subtracting the third equation from the fourth equation gives us $6 = 3x$, so the only possibility for the value of x is $x = 2$. Using $x = 2$, the third equation becomes $5 = 6 + y$, so the only possibility for the value of y is $y = -1$. Also using $x = 2$, the second equation becomes $-8 = -2 - 2z$, so the only possible value of z is $z = 3$.

Therefore, we have determined that the only possible solution to this system of equations is $x = 2$, $y = -1$ and $z = 3$. To see if these values really give a solution, we need to plug them into all four equations and check that they work. I will leave it to you to do this, but you will see that they do work. Therefore, $\langle 9, -8, 5, 11\rangle \in L(S)$ and we can write $\langle 9, -8, 5, 11\rangle$ (uniquely) as a linear combination of the vectors in S as

$$\langle 9, -8, 5, 11\rangle = 2\,\langle 2, -1, 3, 6\rangle - \langle 4, 0, 1, 1\rangle + 3\,\langle 3, -2, 0, 0\rangle$$

Review of indendence

Let $S = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k\}$ be a set of vectors in $\mathbb{R}^n$ and let $\overline{w} \in \mathbb{R}^n$. We say S **spans** $\overline{w}$ **uniquely** if S spans $\overline{w}$ but there is only one way to write $\overline{w}$ as a linear combination of the vectors in S. We say that S is **independent** if s spans $\overline{0}$ uniquely.

Notice that for any nonempty set S of vectors, S spans $\overline{0}$ because we can choose all the scalars in the linear combination to be 0. That is, we always have

$$\overline{0} = 0\,\overline{v}_1 + 0\,\overline{v}_2 + \cdots + 0\,\overline{v}_k$$

So, the question of whether S is independent is really *whether there is any other way to write* $\overline{0}$ *as a linear combination of the vectors in* S.

Before doing some examples, I will remind you of one fact (which is Theorem 12.7 in the textbook, in case you want to review it).

Fact. A set S of vectors spans every vector in $L(S)$ uniquely if and only if S spans $\overline{0}$ uniquely. (That is, if and only if S in independent.)

Example 3. Consider the following set S of vectors in $\mathbb{R}^3$.

$$S = \{\langle 1,2,1\rangle, \langle -1,3,0\rangle, \langle 2,0,1\rangle\}$$

Is S independent?

Solution 3. To check whether S is independent, we need to see whether there are solutions to

$$\langle 0,0,0\rangle = x\,\langle 1,2,1\rangle + y\,\langle -1,3,0\rangle + z\,\langle 2,0,1\rangle$$

in which not all of x, y and z are equal to 0. Rewriting to obtain a system of equations, we get

$$\begin{gathered}\langle 0,0,0\rangle = x\,\langle 1,2,1\rangle + y\,\langle -1,3,0\rangle + z\,\langle 2,0,1\rangle \\ \Leftrightarrow \langle 0,0,0\rangle = \langle x, 2x, x\rangle + \langle -y, 3y, 0\rangle + \langle 2z, 0, z\rangle \\ \Leftrightarrow \langle 0,0,0\rangle = \langle x - y + 2z, 2x + 3y, x + z\rangle\end{gathered}$$

That is, we need to see if there are nonzero solutions to

$$\begin{aligned} 0 &= x - y + 2z \\ 0 &= 2x + 3y \\ 0 &= x + z \end{aligned}$$

The second equation tells us that $y = -2x/3$ and the third equation tells us that $z = -x$. Plugging these values into the first equation gives us

$$0 = x + 2x/3 - 2x$$

so $0 = -x/3$ and we conclude that $x = 0$. Since $y = -2x/3$, we get that $y = 0$, and since $z = -x$, we get that $z = 0$. Therefore, the only solution to

$$\langle 0,0,0\rangle = x\,\langle 1,2,1\rangle + y\,\langle -1,3,0\rangle + z\,\langle 2,0,1\rangle$$

is when $x = y = z = 0$. Hence, S is independent.

Example 4. Consider the following set S of vectors in $\mathbb{R}^4$.

$$S = \{\langle 1,2,0,3\rangle, \langle 2,0,1,4\rangle, \langle 8,4,3,18\rangle\}$$

Is S independent?

Solution 4. As in the last example, we need to see whether there are nonzero solutions to

$$\langle 0,0,0,0\rangle = x\,\langle 1,2,0,3\rangle + y\,\langle 2,0,1,4\rangle + z\,\langle 8,4,3,18\rangle$$

Rewriting, we obtain

$$\begin{aligned} &\langle 0,0,0,0\rangle = x\,\langle 1,2,0,3\rangle + y\,\langle 2,0,1,4\rangle + z\,\langle 8,4,3,18\rangle \\ \Leftrightarrow\ &\langle 0,0,0,0\rangle = \langle x,2x,0,3x\rangle + \langle 2y,0,y,4y\rangle + \langle 8z,4z,3z,18z\rangle \\ \Leftrightarrow\ &\langle 0,0,0,0\rangle = \langle x+2y+8z, 2x+4z, y+3z, 3x+4y+18z\rangle \end{aligned}$$

That is, we need to look for nonzero solutions to

$$\begin{aligned} 0 &= x+2y+8z \\ 0 &= 2x+4z \\ 0 &= y+3z \\ 0 &= 3x+4y+18z \end{aligned}$$

The second equation says that $x = -2z$ and the third equation says that $y = -3z$. If we substitute $-2z$ for x and substitute $-3z$ for y in the first equation, we get

$$0 = -2z + 2(-3z) + 8z$$

which reduces to $0 = 0$. So, the first equation is satisfied by any choice of x, y and z *as long as* $x = -2z$ and $y = -3z$. Similarly, if we do these substitutions into the fourth equation, we have

$$0 = 3(-2z) + 4(-3z) + 18z$$

which also reduces to $0 = 0$. Therefore, the fourth equation is also satisfied by any x, y and z as long as $x = -2z$ and $y = -3z$.

What do we conclude from these calculations? If we choose z to be any real number and then let $x = -2z$ and $y = -3z$, we get a solution to

$$\langle 0,0,0,0\rangle = x\,\langle 1,2,0,3\rangle + y\,\langle 2,0,1,4\rangle + z\,\langle 8,4,3,18\rangle$$

For example, choosing $z = 1$, we have $x = -2$ and $y = -3$, and we know that

$$\langle 0,0,0,0\rangle = -2\,\langle 1,2,0,3\rangle - 3\,\langle 2,0,1,4\rangle + \langle 8,4,3,18\rangle$$

holds. (You can check this to make sure!) Therefore, there are infinitely many ways to write $\overline{0}$ as a linear combination of the vectors in S. So, S is not independent.

Example 5. Consider the following set S of vectors in $\mathbb{R}^3$

$$S = \{\langle 2,1,1\rangle, \langle -3,0,4\rangle, \langle -1,1,5\rangle\}$$

Does S span $\langle 2,4,12\rangle$ uniquely? Is S independent?

Solution 5. To answer the first part of this problem, we need to determine if $\langle 2,4,12\rangle$ is spanned by S, and if so, whether it can be written in more than one way as a linear combination of the vectors in S. In other words, we need to see how many solutions there are to

$$\langle 2,4,12\rangle = x\,\langle 2,1,1\rangle + y\,\langle -3,0,4\rangle + z\,\langle -1,1,5\rangle$$

If there are NO solutions, then S does not span $\langle 2,4,12\rangle$ uniquely (because it does not span it at all). If there is MORE THAN ONE solution, then S does not span $\langle 2,4,12\rangle$ uniquely (because it can be written in more than one way as a linear combination of the vectors in S). And, if there is EXACTLY ONE SOLUTION, then S does span $\langle 2,4,12\rangle$ uniquely.

As usual, we begin by rewriting this equation to get a system of equations.

$$\begin{aligned}
&\langle 2,4,12\rangle = x\,\langle 2,1,1\rangle + y\,\langle -3,0,4\rangle + z\,\langle -1,1,5\rangle\\
\Leftrightarrow\ &\langle 2,4,12\rangle = \langle 2x,x,x\rangle + \langle -3y,0,4y\rangle + \langle -z,z,5z\rangle\\
\Leftrightarrow\ &\langle 2,4,12\rangle = \langle 2x-3y-z, x+z, x+4y+5z\rangle
\end{aligned}$$

So, we need to figure out how many solutions there are to the system of equations

$$\begin{aligned}
2 &= 2x - 3y - z\\
4 &= x + z\\
12 &= x + 4y + 5z
\end{aligned}$$

Adding the first and second equations gives us $6 = 3x - 3y$, or in other words, $2 = x - y$. Thus, we know that $x = 2 + y$.

Similarly, if we multiply the third equation by -2, we get $-24 = -2x - 8y - 10z$. Adding this equation to the first equation gives us $-22 = -11y - 11z$, or in other words, $2 = y + z$. Thus, we know that $z = 2 - y$.

We now have discovered two additional facts – namely, $x = 2+y$ and $z = 2-y$. We might try to more simplifying using other combinations of these equations, but it turns out that we don't need to. Notice what happens when we take our first equation and substitute $2 + y$ for

x and substitute $2-y$ for z.

$$\begin{aligned}2 &= 2x-3y-z\\ 2 &= 2(2+y)-3y-(2-y)\\ 2 &= 4+2y-3y-2+y\\ 2 &= 2\end{aligned}$$

Therefore, this equation is satisfied by any choice of x, y and z *as long as* $x = 2+y$ *and* $z = 2-y$. If you do a similar substitution into the second equation

$$\begin{aligned}4 &= x+z\\ \Leftrightarrow 4 &= (2+y)+(2-y)\\ \Leftrightarrow 4 &= 2+y+2-y\\ \Leftrightarrow 4 &= 4\end{aligned}$$

and for the third equation

$$\begin{aligned}12 &= x+4y+5z\\ \Leftrightarrow 12 &= (2+y)+4y+5(2-y)\\ \Leftrightarrow 12 &= 2+y+4y+10-5y\\ \Leftrightarrow 12 &= 12\end{aligned}$$

Therefore, all three equations are satisfied by any choice of x, y and z for which $x = 2+y$ and $z = 2-y$. What does this calculation tell us? It means that as long as $x = 2+y$ and $z = 2-y$, the relationship

$$\langle 2,4,12\rangle = x\,\langle 2,1,1\rangle + y\,\langle -3,0,4\rangle + z\,\langle -1,1,5\rangle$$

holds. That is, we can pick any real value for y whatsoever and then let $x = 2+y$ and $z = 2-y$ and we get scalars x, y and z for which

$$\langle 2,4,12\rangle = x\,\langle 2,1,1\rangle + y\,\langle -3,0,4\rangle + z\,\langle -1,1,5\rangle$$

In other words, there are infinitely many ways to write $\langle 2,4,12\rangle$ as a linear combination of $\langle 2,1,1\rangle$, $\langle -3,0,4\rangle$ and $\langle -1,1,5\rangle$! That means that $\langle 2,4,12\rangle$ is not uniquely spanned by S. So, we have answered the first part of this problem.

We have actually answered the second part of the problem as well because we can immediately conclude that S is not independent. Why? By the Fact at this beginning of this section, we know that if S were linearly independent, then S would uniquely span each vector in $L(S)$. But, $\langle 2,4,12\rangle$ is in $L(S)$. So, if S were independent, then S would uniquely span $\langle 2,4,12\rangle$. Since S does NOT uniquely span this vector, S cannot be independent.

Review of bases

A set of vectors $S = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k\}$ in $\mathbb{R}^n$ is a **basis for** $\mathbb{R}^n$ if S spans every vector in $\mathbb{R}^n$ uniquely. Unpacking the meaning of this definition may at first seem daunting, but there is a nice geometric way to think about a basis and a nice calculational tool for determining if S is a basis.

First, the geometric point of view. I'll speak about $\mathbb{R}^3$ to make things concrete. The standard basis for $\mathbb{R}^3$ is the set of vectors

$$\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle$$

Notice that for any vector $\langle a,b,c\rangle$ in $\mathbb{R}^3$, the only way to form $\langle a,b,c\rangle$ as a linear combination of these three vectors is by

$$\langle a,b,c\rangle = a\,\langle 1,0,0\rangle + b\,\langle 0,1,0\rangle + c\,\langle 0,0,1\rangle$$

So, the standard basis really is a basis. Geometrically, when we write a vector like $\langle 3,4,-2\rangle$, we are thinking that $\langle 3,4,-2\rangle$ is formed from 3 copies of $\langle 1,0,0\rangle$, 4 copies of $\langle 0,1,0\rangle$ and -2 copies of $\langle 0,0,1\rangle$. That is, we can describe the vector $\langle 3,4,-2\rangle$ uniquely in terms of the standard "coordinate directions" $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$ and $\langle 0,0,1\rangle$. But, there is no reason why we should have to choose the vectors $\langle 1,0,0\rangle$, $\langle 0,1,0\rangle$ and $\langle 0,0,1\rangle$ to give out coordinates!

The geometric point of a basis is that it gives an alternate "coordinate" description of $\mathbb{R}^n$. That is, if S is a basis for $\mathbb{R}^n$, then every vector in $\mathbb{R}^n$ (i.e. every direction in $\mathbb{R}^n$) can be uniquely decomposed into the "coordinate" directions in S. That is the geometric picture you should have in mind.

Second, the calculational picture. How do we tell if S is a basis? Well, based on the definition, we need to check two things

- Does S span every vector in $\mathbb{R}^n$? In other words, can every vector in $\mathbb{R}^n$ be written as a linear combination of the vectors in S? (This part of the definition is requiring that S gives enough "coordinate directions" to form every vector.)
- Does S span every vector in $\mathbb{R}^n$ *uniquely*? (This part of the definition is making sure that S is not trying to use too many "coordinate directions".) If we have already answered the first question, namely that S spans all the vectors in $\mathbb{R}^n$, then we know $L(S) = \mathbb{R}^n$. So, asking if S spans the vectors in $\mathbb{R}^n$ uniquely reduces to just asking is S is independent.

Answering these questions may still seem daunting, but we proved a theorem last semester (Theorem 12.10 in the book) that simplifies things drastically. We proved that the dimension of $\mathbb{R}^n$ in terms of the number of "coordinate directions" in any basis is always n. That is, there is no way to describe a "coordinate system" for $\mathbb{R}^n$ that uses more or less than n "coordinate directions". In practical terms, we proved that a basis for $\mathbb{R}^n$ always contains exactly n vectors! So, to determine if S is a basis for $\mathbb{R}^n$, we can use the following trick.

Fact. A set of vectors S in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$ if and only if S is linearly independent and contains exactly n vectors.

Example 6. Is the set S a basis for $\mathbb{R}^4$?

$$S = \{\langle 1,0,1,2\rangle, \langle 3,-2,1,0\rangle, \langle -1,1,0,2\rangle\}$$

Solution 6. No! S only contains 3 vectors, so it cannot be a basis for $\mathbb{R}^4$.

Example 7. Is the set S a basis for $\mathbb{R}^2$?

$$S = \{\langle 1,3\rangle, \langle 2,-4\rangle\}$$

Solution 7. Since we are working in $\mathbb{R}^2$ and S contains 2 vectors, it suffices to check whether S is independent. So, we need to look for nonzero solutions to

$$\langle 0,0\rangle = x\,\langle 1,3\rangle + y\,\langle 2,-4\rangle$$

As usual, we turn this equation into a system of linear equation. I'll let you work through the steps, but you should end up with

$$\begin{aligned} 0 &= x + 2y \\ 0 &= 3x - 4y \end{aligned}$$

It is straightforward to solve this system and see that the only solution is $x = 0$ and $y = 0$. Therefore, S is independent and forms a basis for $\mathbb{R}^2$.