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Week 14 Problem Solutions

Problem 1 $\bar{r}(t) = \langle 3t - t^3,\ 3t^2,\ 3t + t^3 \rangle \Rightarrow \boxed{\bar{r}(1) = \langle 2, 3, 4 \rangle}$

$\bar{v}(t) = \langle 3 - 3t^2,\ 6t,\ 3 + 3t^2 \rangle \Rightarrow \boxed{\bar{v}(1) = \langle 0, 6, 6 \rangle}$

$\|\bar{v}(t)\| = \sqrt{(3-3t^2)^2 + (6t)^2 + (3+3t^2)^2} = \sqrt{9 - 18t^2 + 9t^4 + 36t^2 + 9 + 18t^2 + 9t^4}$

$\|\bar{v}(t)\| = \sqrt{18t^4 + 36t^2 + 18} = \sqrt{18(t^4 + 2t^2 + 1)} = \sqrt{18} \cdot \sqrt{(t^2+1)^2}$

$\Rightarrow \|\bar{v}(t)\| = 3\sqrt{2}(t^2+1) \Rightarrow \boxed{\|\bar{v}(1)\| = 6\sqrt{2}}$ (which you could also get from above)

$\bar{T}(t) = \dfrac{\bar{v}(t)}{\|\bar{v}(t)\|} = \dfrac{1}{3\sqrt{2}(t^2+1)} \langle 3 - 3t^2,\ 6t,\ 3 + 3t^2 \rangle$

$\bar{T}(t) = \dfrac{1}{\sqrt{2}} \left\langle \dfrac{1-t^2}{t^2+1},\ \dfrac{2t}{t^2+1},\ 1 \right\rangle \Rightarrow \boxed{\bar{T}(1) = \dfrac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle}$

$\bar{a}(t) = \langle -6t, 6, 6t \rangle \Rightarrow \boxed{\bar{a}(1) = \langle -6, 6, 6 \rangle}$

$$\bar{T}'(t) = \frac{1}{\sqrt{2}} \left\langle \frac{(t^2+1)(-2t) - (1-t^2)(2t)}{(t^2+1)^2},\ \frac{(t^2+1)(2) - (2t)(2t)}{(t^2+1)^2},\ 0 \right\rangle$$

$$= \frac{1}{\sqrt{2}} \left\langle \frac{-2t^3 - 2t - 2t + 2t^3}{(t^2+1)^2},\ \frac{2t^2 + 2 - 4t^2}{(t^2+1)^2},\ 0 \right\rangle$$

$$\bar{T}'(t) = \frac{1}{\sqrt{2}} \left\langle \frac{-4t}{(t^2+1)^2},\ \frac{2 - 2t^2}{(t^2+1)^2},\ 0 \right\rangle$$

To find $\bar{N}(1)$, $\bar{T}'(1) = \frac{1}{\sqrt{2}}\langle -1,0,0\rangle$ and $\bar{N}(1) = \frac{\bar{T}'(1)}{\|\bar{T}'(1)\|}$.

Since $\|\bar{T}'(1)\| = \frac{1}{\sqrt{2}}$, we get $\boxed{\bar{N}(1) = \langle -1,0,0\rangle}$.

The osculating plane goes through $\bar{r}(1) = \langle 2,3,4\rangle$ with direction vectors $\bar{T}(1) = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$ and $\bar{N}(1) = \langle -1,0,0\rangle$.

In vector form: $\langle 2,3,4\rangle + t\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle + s\langle -1,0,0\rangle$

In parametric form: $x = 2 - s$, $y = 3 + \frac{t}{\sqrt{2}}$, $z = 4 + \frac{t}{\sqrt{2}}$

For normal form, find $\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle \times \langle -1,0,0\rangle$

$$\langle 0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle \times \langle -1,0,0\rangle = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ -1 & 0 & 0 \end{vmatrix} = \left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Normal form: $\langle x,y,z\rangle \cdot \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle = \langle 2,3,4\rangle \cdot \langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

$$\Rightarrow \; -\frac{y}{\sqrt{2}} + \frac{z}{\sqrt{2}} = -\frac{3}{\sqrt{2}} + \frac{4}{\sqrt{2}} \;\Rightarrow\; \boxed{-y + z = 1}$$

For decomposition of $\bar{a}(t)$: $\bar{a}(t) = \|\bar{v}(t)\|'\,\bar{T}(t) + \|\bar{v}(t)\|\,\|\bar{T}'(t)\|\,\bar{N}(t)$

$\|\bar{v}(t)\| = 3\sqrt{2}(t^2+1) \Rightarrow \|\bar{v}(t)\|' = 6\sqrt{2}t \Rightarrow \|\bar{v}(1)\|' = 6\sqrt{2}$

$\|\bar{v}(1)\| = 3\sqrt{2}(1^2+1) \Rightarrow \|\bar{v}(1)\| = 6\sqrt{2}$. Therefore

$$\langle -6,6,6\rangle = \bar{a}(1) = 6\sqrt{2}\left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle + (6\sqrt{2})\left(\frac{1}{\sqrt{2}}\right)\langle -1,0,0\rangle$$

$$\Rightarrow \; \bar{a}(1) = 6\sqrt{2}\,\bar{T}(1) + 6\,\bar{N}(1)$$

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Problem 2

- The x-y components trace out a circle of radius a. The z component moves upwards, so the graph is a helix in which position rises $2\pi b$ units each time x-y coordinates complete a circle.

- $\boxed{\bar{v}(t) = \langle -a\sin t, a\cos t, b\rangle}$, $\boxed{\bar{a}(t) = \langle -a\cos t, -a\sin t, 0\rangle}$

$$\|\bar{v}(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \Rightarrow \|\bar{v}\|^3 = (a^2+b^2)^{3/2}$$

$$\bar{v}\times\bar{a} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \langle ab\sin t, -ab\cos t, \overbrace{a^2\sin^2 t + a^2\cos^2 t}^{a^2}\rangle$$

$$\|\bar{v}\times\bar{a}\| = \sqrt{a^2b^2\sin^2 t + a^2b^2\cos^2 t + (a^2)^2} = \sqrt{a^2b^2+a^4} = a\sqrt{b^2+a^2}$$

$$\Rightarrow \frac{\|\bar{v}\times\bar{a}\|}{\|\bar{v}\|^3} = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}$$

- $\boxed{\bar{T}(t) = \frac{1}{\sqrt{a^2+b^2}}\langle -a\sin t, a\cos t, b\rangle}$

$$\bar{T}'(t) = \frac{1}{\sqrt{a^2+b^2}}\cdot\langle -a\cos t, -a\sin t, 0\rangle$$

$$\|\bar{T}'(t)\| = \frac{1}{\sqrt{a^2+b^2}}\sqrt{\underbrace{a^2\cos^2 t + a^2\sin^2 t}_{a^2}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\overline{N}(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{\sqrt{a^2+b^2}}{a} \cdot \frac{1}{\sqrt{a^2+b^2}} \langle -a\cos t, -a\sin t, 0 \rangle$$

$$\Rightarrow \boxed{\overline{N}(t) = \langle -\cos t, -\sin t, 0 \rangle}$$ To show $\overline{T}(t) \perp \overline{N}(t)$:

$$\overline{T}(t) \cdot \overline{N}(t) = \frac{1}{\sqrt{a^2+b^2}} \langle -a\sin t, a\cos t, b \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle$$

$$= \frac{1}{\sqrt{a^2+b^2}} \left(a\sin t\cos t - a\cos t\sin t + 0 \right) = 0.$$

At $t = \pi$: $\overline{r}(\pi) = \langle -a, 0, b\pi \rangle$, $\overline{T}(\pi) = \left\langle 0, \frac{-a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$

$\overline{N}(\pi) = \langle 1, 0, 0 \rangle$.

$$\overline{T}(\pi) \times \overline{N}(\pi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & \frac{-a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ 1 & 0 & 0 \end{vmatrix} = \left\langle 0, \frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}} \right\rangle$$

$$\langle x, y, z \rangle \cdot \langle -a, 0, b\pi \rangle = \langle -a, 0, b\pi \rangle \cdot \left\langle 0, \frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}} \right\rangle$$

$$\boxed{-ax + b\pi z = \frac{ab}{\sqrt{a^2+b^2}} \quad \text{normal form for oscillating plane}}$$

$\|\overline{v}(t)\| = \sqrt{a^2+b^2}$ is constant, so $\|\overline{v}(t)\|' = 0$. Therefore the tangential component of acceleration is 0. The normal component of acceleration is $\|\overline{v}(t)\| \, \|\overline{T}'(t)\|$ which is $\sqrt{a^2+b^2} \cdot \frac{a}{\sqrt{a^2+b^2}} = a$. You can also see this from formular above because $\overline{a}(t) = a\overline{N}(t)$.

(3) $\boxed{\text{Problem 3}}$ $\|\bar{v}(t)\|^2 = \bar{v}(t)\cdot\bar{v}(t)$, so

$$\frac{d}{dt}\|\bar{v}(t)\|^2 = \frac{d}{dt}\left(\bar{v}(t)\cdot\bar{v}(t)\right) = \bar{v}'(t)\cdot\bar{v}(t) + \bar{v}(t)\cdot\bar{v}'(t) = 2\bar{v}(t)\cdot\overset{\bar{a}(t)}{\overset{\downarrow}{\bar{v}'(t)}}$$

$$\Rightarrow \frac{d}{dt}\|\bar{v}(t)\|^2 = 2\bar{v}(t)\cdot\bar{a}(t).$$

$\boxed{\text{Problem 4}}$ $\boxed{4(a)}$ $\bar{v}(t)$ is not parallel to $\bar{r}(t)$ means $\bar{v}(t)\neq c\bar{r}(t)$ for any $c\in\mathbb{R}$. Therefore $\{\bar{r}(t),\bar{v}(t)\}$ is independent, so $\bar{r}(t)\times\bar{v}(t)\neq\bar{0}$.
Since $\bar{a}(t) = -k\bar{r}(t)$, we have $\bar{a}(t)+k\bar{r}(t)=\bar{0}$ so $\{\bar{r}(t),\bar{a}(t)\}$ is linearly dependent. Therefore, $\bar{r}(t)\times\bar{a}(t)=\bar{0}$.

$\boxed{4(b)}$ $\frac{d}{dt}\left[\bar{r}(t)\times\bar{v}(t)\right] = \left(\bar{r}'(t)\times\bar{v}(t)\right)+\left(\bar{r}(t)\times\bar{v}'(t)\right)$

$$= \left(\bar{v}(t)\times\bar{v}(t)\right)+\left(\bar{r}(t)\times\bar{a}(t)\right) = \bar{0}+\bar{0}=\bar{0}.$$

Since $\left(\bar{r}(t)\times\bar{v}(t)\right)'=\bar{0}$ we have $\bar{r}(t)\times\bar{v}(t)=\bar{c}$ for some constant vector $\bar{c}$.

$\boxed{4(c)}$ Since $\bar{r}(t)\times\bar{v}(t)=\bar{c}$ we know that $\bar{r}(t)\perp\bar{c}$ for all t. Therefore, $\bar{r}(t)$ lies in a plane with normal vector $\bar{c}$ (and contains the point $\bar{r}(0)$, or any other specific value $\bar{r}(t_0)$ for fixed t_0).

$\boxed{\text{Problem 5}}$ $\bar{v}(t) = \langle\sqrt{2}, e^t, -e^{-t}\rangle \Rightarrow \|\bar{v}\| = \sqrt{2+(e^t)^2+(-e^{-t})^2}$

$$\Rightarrow \|\bar{v}(t)\| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t+e^{-t}$$

$$\text{Arclength} = \int_0^1 e^t+e^{-t}\,dt = e^t-e^{-t}\Big|_0^1 = (e^1-e^{-1})-(e^0-e^0) = \boxed{e-\frac{1}{e}}$$

[Skip Problem 6 - not covered on exam.]

$\boxed{\text{Problem 7}}$ $\boxed{7(a)}$ We know $\overset{\bar{r}''(t)}{\overset{\shortparallel}{\bar{a}(t)}} = \|\bar{v}(t)\|'\,\bar{T}(t) + \|\bar{v}(t)\|\,\|\bar{T}'(t)\|\,N(t)$.

Since $\bar{v}(t) = \bar{r}'(t)$ this shows $a_T(t) = \|\bar{r}'(t)\|'$. For $a_N(t)$, take cross product of both sides with $\bar{v}(t)$, or equivalently $\bar{r}'(t)$:

$$\underset{\bar{v}(t)}{\underset{\uparrow}{\bar{r}'(t)}} \times \underset{\bar{a}(t)}{\underset{\uparrow}{\bar{r}''(t)}} = a_T(t)\cdot\big(\underbrace{\bar{r}'(t)\times\bar{T}(t)}_{\bar{0}\text{ b/c } \bar{r}'(t) \text{ and } \bar{T}(t) \text{ point in same direction.}}\big) + \underset{\substack{\text{we know } a_N(t)\geq 0 \text{ by} \\ \text{formula above} \\ \text{so no absolute values here}}}{\underset{\uparrow}{a_N(t)}}\big(\bar{v}(t)\times\bar{N}(t)\big)$$

$$\Rightarrow \|\bar{r}'(t)\times\bar{r}''(t)\| = a_N(t)\,\|\bar{v}(t)\times\bar{N}(t)\| = a_N(t)\,\|\bar{v}(t)\|\,\underbrace{\|\bar{N}(t)\|}_{1}\sin\theta$$

and $\sin\theta = 1$ b/c $\bar{v}(t)\perp\bar{N}(t)$ since $\bar{v}(t)$ in same direction as $\bar{T}(t)$,

$$\Rightarrow \|\bar{r}'(t)\times\bar{r}''(t)\| = a_N(t)\cdot\|\bar{v}(t)\| \Rightarrow a_N(t) = \frac{\|\bar{r}'(t)\times\bar{r}''(t)\|}{\|\bar{v}(t)\|}$$

$\boxed{7(b)}$ $\bar{v}(t) = \langle 2t, 2t, 3t^2\rangle \Rightarrow \bar{a}(t) = \langle 2, 2, 6t\rangle$

$$\|\bar{v}(t)\| = \sqrt{4t^2+4t^2+9t^4} = \sqrt{9t^4+8t^2} \Rightarrow \|\bar{v}(t)\|' = \frac{36t^3+16t}{2\sqrt{9t^4+8t^2}}$$

$$\boxed{\text{Simplifying, } a_T(t) = \|\bar{v}\|' = \frac{18t^2+8}{\sqrt{9t^2+8}} \quad \text{(tangential component of } \bar{a}(t))}$$

$$\bar{v}(t)\times\bar{a}(t) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t^2, 0\rangle, \quad \|\bar{v}(t)\times\bar{a}(t)\| = \sqrt{36t^4+36t^4} = 6\sqrt{2}\,t^2$$

$$\boxed{\therefore\ a_N(t) = \frac{6\sqrt{2}\,t^2}{\sqrt{9t^4+8t^2}} = \frac{6\sqrt{2}\,t}{\sqrt{9t^2+8}} \quad \text{Normal component of } \bar{a}(t)}$$

$\boxed{\text{SKIP 7(c) and Problem 8}}$