Math 2142 Exam 1 Review Problems

Problem 1. Calculate the 3rd Taylor polynomial for $\arcsin x$ at $x = 0$.

Solution. Let $f(x) = \arcsin x$. For this problem, we use the formula

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3$$

for the 3rd Taylor polynomial at $x = 0$. To calculate the various quantities:

$$\begin{aligned} f(x) = \arcsin x \quad &\Rightarrow \quad f(0) = 0 \\ f'(x) = (1 - x^2)^{-1/2} \quad &\Rightarrow \quad f'(0) = 1 \\ f''(x) = x(1 - x^2)^{-3/2} \quad &\Rightarrow \quad f''(0) = 0 \\ f'''(x) = (1 - x^2)^{-3/2} + 3x(1 - x^2)^{-5/2} \quad &\Rightarrow \quad f'''(0) = 1 \end{aligned}$$

Therefore, the 3rd Taylor polynomial is $x + x^3/6$.

Problem 2. In this problem, you will calculate the n-th Taylor polynomial for $x^{-1/2}$ at $x = 1$. To make the notation easier, let $f(x) = x^{-1/2}$.

2(a). Prove by induction that for all $n \geq 1$,

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n}\,x^{-(2n+1)/2}$$

Solution. The base case for the induction is when $n = 1$. In this case,

$$f'(x) = -\frac{1}{2}x^{-3/2} = (-1)^1 \frac{1}{2^1} x^{-(2 \cdot 1+1)/2}$$

so the formula works.

For the induction case, assume that $f^{(n)}(x)$ has the form above and we show that $f^{(n+1)}(x)$ also has this form. Since $f^{(n+1)}(x) = d/dx\, f^{(n)}(x)$, we can take the derivative of the formula above to get

$$f^{(n+1)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \cdot \frac{-(2n+1)}{2} x^{(-(2n+1)/2)-1}$$

I will leave you to check that this expression simplifies in the correct form of

$$f^{(n+1)}(x) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2(n+1)-1)}{2^{n+1}}\,x^{-(2(n+1)+1)/2}$$

2(b). Give the formula for the n-th Taylor polynomial to $x^{-1/2}$ at $x = 1$.

Solution. We have $f(1) = 1$ and by Problem 2(a), we know

$$f^{(k)}(1) = (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k}$$

for $k \geq 1$. Therefore, the n-th Taylor polynomial for $f(x)$ at $x = 1$ is

$$1 + \sum_{k=1}^{n} (-1)^k \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2^k \cdot k!} (x-1)^k$$

or written without the sum notation

$$1 - \frac{1}{2}(x-1) + \frac{1 \cdot 3}{2^2 \cdot 2!}(x-1)^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}(x-1)^3 + \cdots + (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}(x-1)^n$$

Problem 3. In this problem, you will develop the Taylor polynomials for $\ln(1-x)$. Let $f(x) = \ln(1-x)$.

3(a). Prove by induction that for all $n \geq 1$,

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}$$

and hence for $n \geq 1$, $f^{(n)}(0) = -(n-1)!$.

Solution. For the base case when $n = 1$, we need to show that

$$f'(x) = -\frac{0!}{(1-x)^1}$$

By the Chain Rule, $d/dx(\ln(1-x)) = -1/(1-x)$. Since $0! = 1$, the right side of this above equation is also $-1/(1-x)$. This completes the base case.

For the induction case, assume that

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}$$

for a fixed $n \geq 1$. We need to show that

$$f^{(n+1)}(x) = -\frac{((n+1)-1)!}{(1-x)^{n+1}}$$

The right side is $-n!/(1-x)^{n+1}$. To calculate the left side using the induction hypothesis

$$f^{(n+1)}(x) = \frac{d}{dx} f^{(n)}(x) = \frac{d}{dx}\left(-(n-1)!\,(1-x)^{-n}\right) = -(n-1)!(-n)(-1)(1-x)^{-(n+1)}$$

The last expression simplifies to $-n!/(1-x)^{(n+1)}$ as required.

3(b). Give the n-th Taylor polynomial for $f(x)$ at $x = 0$.

Since $f(0) = \ln 1 = 0$, we know that the constant term in the Taylor polynomial is 0. So, we can start the indexing in our sum with $k = 1$ (because the $k = 0$ term is 0). By Problem 3(a), we know that $f^{(n)}(0) = -(n-1)!$ for $n \geq 1$. Therefore, the n-th Taylor polynomial for $\ln(1-x)$ at $x = 0$ is

$$T_n = \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!}\, x^k = \sum_{k=1}^{n} \frac{-(k-1)!}{k!}\, x^k = \sum_{k=1}^{n} -\frac{x^k}{k}$$

Problem 4. Prove that if c is a constant and f is a (sufficiently differentiable) function, then

$$T_n(cf(x)) = c\,T_n(f(x))$$

where we take the Taylor polynomial at $x = a$.

Solution. We first calculate the right side as follows

$$c\,T_n(f(x)) = c \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k = \sum_{k=0}^{n} \frac{c\,f^{(k)}(a)}{k!}\,(x-a)^k$$

To calculate the left side, we first note that because c is a constant, we have

$$\frac{d^k}{dx^k}\,cf = c\,\frac{d^k}{dx^k}\,f = c\,f^{(k)}$$

We now calculate the left side as follows.

$$T_n(cf(x)) = \sum_{k=0}^{n} \frac{(d^k/dx^k\,cf)(a)}{k!}\,(x-a)^k = \sum_{k=0}^{n} \frac{c\,f^{(k)}(a)}{k!}\,(x-a)^k$$

As these formulas now match, we have finished our proof.

Problem 5. Evaluate the following (convergent) improper integrals.

For the first integral,

$$\int_4^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} \int_4^t \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty} \left.\frac{-1}{3(3x+1)}\right|_4^t = \lim_{t\to\infty} \frac{-1}{9t+3} + \frac{1}{3\cdot 13} = \frac{1}{39}$$

For the second integral,

$$\int_2^\infty e^{-x/2}\,dx = \lim_{t\to\infty} \int_2^t e^{-x/2}\,dx = \lim_{t\to\infty} \left. -2e^{-x/2}\right|_2^t = \lim_{t\to\infty} -2e^{-t/2} + 2e^{-1} = 2/e$$

For the third integral, we need to split the integral into two improper integrals by picking a convenient middle point. I'll choose to split it as

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{-|x|}\,dx + \int_{0}^{\infty} e^{-|x|}\,dx$$

We need to make sure that both of the integrals on the right converge and sum their values. First, consider

$$\int_{0}^{\infty} e^{-|x|}\,dx = \lim_{t\to\infty} \int_{0}^{t} e^{-x}\,dx = \lim_{t\to\infty} -e^{-x}\Big|_{0}^{t} = \lim_{t\to\infty} -e^{-t} + e^{0} = 1$$

Notice that we were able to remove the absolute value sign because x is positive on the interval $(0, \infty]$. You can approach the other integral in a couple of different ways. One method is to note that by symmetry, the area under $e^{-|x|}$ on the interval $(-\infty, 0)$ is the same as on the interval $(0, \infty)$. Therefore, we must have $\int_{-\infty}^{0} e^{-|x|}\,dx = 1$ as well. Alternately, you can write out the definition and calculate the integral directly. However, remember that $|x| = -x$ on the interval $(-\infty, 0)$!

$$\int_{-\infty}^{0} e^{-|x|}\,dx = \lim_{t\to-\infty} \int_{t}^{0} e^{-(-x)}\,dx = \lim_{t\to-\infty} \int_{t}^{0} e^{x}\,dx = \lim_{t\to-\infty} e^{x}\Big|_{t}^{0} = \lim_{t\to-\infty} e^{0} - e^{t} = 1$$

Problem 6. Use the definition of the improper integral to explain why $\int_0^\infty \sin x\,dx$ diverges.

Solution. By definition,

$$\int_{0}^{\infty} \sin x\,dx = \lim_{t\to\infty} \int_{0}^{t} \sin x\,dx$$

provided this limit exists. To show that the limit does not exist, we note that

$$\int_{0}^{t} \sin x\,dx = -\cos x\Big|_{0}^{t} = -\cos t + \cos 0 = -\cos t + 1$$

Recall that when we plug an even multiple of π into $\cos t$ we get 1. That is, $\cos 2n\pi = 1$. When we plug an odd multiple of π into $\cos t$, we get -1. That is, $\cos(2n+1)\pi = -1$. This means that

$$\int_{0}^{2n\pi} \sin x\,dx = 0 \qquad \text{and} \qquad \int_{0}^{2(n+1)\pi} \sin x\,dx = 2$$

Therefore, for every $M > 0$, there are numbers $t_0 > M$ such that $\int_0^{t_0} \sin x\,dx = 0$ and $t_1 > M$ such that $\int_0^{t_1} \sin x\,dx = 2$. Therefore, $\int_0^t \sin x\,dx$ cannot approach a limit as $t \to \infty$.

Problem 7. Determine whether the following integrals converge or diverge. You do not need to calculate the value of the convergent integrals.

For the first integral

$$\int_{1}^{\infty} \frac{x^2}{9 + x^6}\,dx$$

we can use the Comparison Test as follows:

$$\frac{x^2}{9+x^6} \leq \frac{x^2}{x^6} = \frac{1}{x^4}$$

for all $x \in [1,\infty)$ because $9+x^6 \geq x^6$. We know $\int_1^\infty 1/x^4\,dx$ converges because it has the form $\int_1^\infty 1/x^p\,dx$ with $p=4>1$. Therefore, the first integral converges.

For the second integral,

$$\int_1^\infty \frac{2+e^{-x}}{x}\,dx$$

we can use the Comparison Test as follows:

$$\frac{2+e^{-x}}{x} \geq \frac{1}{x}$$

because $2+e^{-x} \geq 1$. We know $\int_1^\infty 1/x\,dx$ diverges from class, and so the second integral diverges.

For the third integral,

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$$

the problematic integral limit is the lower limit of 0. For $x \in [0,1]$, we know that e^{-x} is between e^{-1} and 1. Therefore, for $x \in (0,1)$, we have

$$\frac{1}{e\sqrt{x}} \leq \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

We know that $\int_0^1 1/\sqrt{x}\,dx$ converges from Homework 3 since it has the form $\int_0^1 1/x^p\,dx$ where $p=1/2<1$. Therefore, the second inequality above tells use that $\int_0^1 e^{-x}/\sqrt{x}\,dx$ converges as well by the Comparison Test.

For the fourth integral,

$$\int_2^\infty \frac{\ln x}{x^2}\,dx$$

we might be tempted to try the Comparison Test with $1/x^2$. However, the inequality goes the wrong way. That is,

$$\frac{1}{x^2} \leq \frac{\ln x}{x^2}$$

but $\int_1^\infty 1/x^2\,dx$ converges, so the Comparison Test doesn't help using this particular comparison and we need to think a little more. We know that as $x \to \infty$, $\ln x$ goes to infinity more slowly than any positive power of x, so our intuition says that the factor of $\ln x$ in the numerator should not be enough to push the integral into divergence.

To see this more formally, try the Comparison Test with $1/x^p$ where p is a little smaller than 2 rather than with $1/x^2$. For example, it should be that for large values of x, we have $\ln x \leq \sqrt{x}$. Since

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$$

we can apply the definition of the limit to fix an M such that for all $x > M$, $\ln x/\sqrt{x} < 1$. In other words, $\ln x < \sqrt{x}$ for all $x > M$. By choosing M larger if necessary, we can assume that $M \geq 2$.

We now have that for all $x > M$,

$$\frac{\ln x}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}$$

This comparison is useful since we know $\int_M^\infty 1/x^{3/2}\,dx$ converges because $p = 3/2 > 1$. To finish the problem, we write

$$\int_2^\infty \frac{\ln x}{x^2}\,dx = \int_2^M \frac{\ln x}{x^2}\,dx + \int_M^\infty \frac{\ln x}{x^2}\,dx$$

The first integral on the right side exists because the function being integrated is continuous and the second integral on the right side converges by the Comparison Test. Therefore, our initial integral converges as well.

For the last integral

$$\int_1^\infty \frac{x}{1+x^2}\,dx$$

we might be tempted to try a comparison with $1/x$. However,

$$\frac{x}{1+x^2} \leq \frac{x}{x^2} = \frac{1}{x}$$

Since $\int_1^\infty 1/x\,dx$ diverges, this comparison doesn't help us, and as above, we need to think a little more. Our intuition should be that for large values of x, $x/(1+x^2)$ is extremely close to x/x^2 so adding the constant 1 should make the fraction small enough to push the integral into convergence. One way to make this more formal is to notice that for $x \geq 1$, we have $1 \leq x^2$ and hence $1 + x^2 \leq x^2 + x^2 = 2x^2$. Since make the denominator larger will make a fraction smaller, this insight tells us that

$$\frac{x}{1+x^2} \geq \frac{x}{2x^2} = \frac{1}{2x}$$

for all $x \geq 1$. Now our comparison goes the right way! We know $\int_1^\infty 1/x\,dx$ diverges, and hence so does $\int_1^\infty 1/2x\,dx$. So, the Comparison Test tells us that this final integral diverges as well.

Problem 8. Find a general formula to write $(a + ib)^{-1}$ in the form $c + id$. That is, find formulas for c and d in terms of a and b. (You can assume $a + ib$ is not 0.)

Solution. To find the multiplicative inverse of $a + ib$ we calculate

$$\frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\,\frac{b}{a^2+b^2}$$

Problem 9. Prove that e^z is not equal to 0 for any $z \in \mathbb{C}$.

Solution. Write $z = a + ib$ where a and b are real numbers. We know

$$e^z = e^{a+ib} = e^a \cos b + i\, e^a \sin b$$

To prove that this value can never be 0, we need to show that there are no real numbers a and b such that both $e^a \cos b = 0$ and $e^a \sin b = 0$. Since $e^a \neq 0$ (because a is real), it suffices to show that we cannot have both $\sin b = 0$ and $\cos b = 0$. If $\sin b = 0$, then b is a integer multiple of π. However, the value of $\cos b$ when b is an integer value of π is either 1 or -1. Therefore, when $\sin b = 0$, we have $\cos b \neq 0$, and hence they cannot both be 0 at the same time.

Problem 10. Find the following limits.

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} \qquad \lim_{x\to 0} \frac{\tan 2x}{\sin 3x} \qquad \lim_{x\to 0} \frac{\sin x}{\arctan x}$$

$$\lim_{x\to\infty} \frac{2^x}{3^x} \qquad \lim_{x\to 0} x^x$$

$$\lim_{x\to 1^-} x^{1/(1-x)} \qquad \lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x - 1}$$

Solution. We consider the limits one at a time, typically using L'Hopital's rule. The first three limits all have form $0/0$, so we immediately apply L'Hopital's rule. In all three cases, we only need to apply it once.

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$$

$$\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} = \lim_{x\to 0} \frac{2\sec^2 2x}{3\cos 3x} = \frac{2}{3}$$

$$\lim_{x\to 0} \frac{\sin x}{\arctan x} = \lim_{x\to 0} \frac{\cos x}{1/(1+x^2)} = 1$$

There are a couple of ways you could approach the next limit. One method would be to simplify $2^x/3^x$ to $(2/3)^x$. Since $0 < 2/3 < 1$, the value of $(2/3)^x$ goes to 0 as the exponent goes to ∞. That is,

$$\lim_{x\to\infty} \frac{2^x}{3^x} = \lim_{x\to\infty} (2/3)^x = 0.$$

Alternately, you can rewrite 2^x as $e^{x\ln 2}$ and 3^x as $e^{x\ln 3}$. Then

$$\lim_{x\to\infty}\frac{2^x}{3^x} = \lim_{x\to\infty}\frac{e^{x\ln 2}}{e^{x\ln 3}} = \lim_{x\to\infty} e^{x(\ln 2-\ln 3)}$$

Since $\ln 2 - \ln 3$ is negative, the exponent of e goes to $-\infty$ as x goes to ∞. Therefore, the limit is 0.

The next limit has for 0^0, so we start by rewriting x^x.

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{\ln x^x} = \lim_{x\to 0} e^{x\ln x}$$

We need to find the limit of $x\ln x$, which has the form $0\cdot\infty$, so we rewrite it in the form ∞/∞ as follows.

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} = \lim_{x\to 0}\frac{1/x}{-1/x^2} = \lim_{x\to 0} -x = 0$$

so $\lim_{x\to 0} x^x = e^0 = 1$.

The next limit has form 1^∞, so we start by rewriting again.

$$\lim_{x\to 1^-} x^{1/(1-x)} = \lim_{x\to 1^-} e^{\ln x^{1/(1-x)}} = \lim_{x\to 1^-} e^{\frac{\ln x}{1-x}}$$

The limit of the exponent has form 0/0, so we apply L'Hopital's rule:

$$\lim_{x\to 1^-}\frac{\ln x}{1-x} = \lim_{x\to 1^-}\frac{1/x}{-1} = \lim_{x\to 1^-} -1/x = -1$$

So, the original limit is e^{-1}, or equivalently $1/e$.

The next limit has form $\infty-\infty$, so we combine it into one fraction.

$$\lim_{x\to 0}\frac{1}{x} - \frac{1}{e^x-1} = \lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)} = \lim_{x\to 0}\frac{e^x-1-x}{xe^x-x}$$

The limit now has the form 0/0 so we try L'Hopital's rule.

$$\lim_{x\to 0}\frac{e^x-1-x}{xe^x-x} = \lim_{x\to 0}\frac{e^x-1}{e^x+xe^x-1}$$

This limit still has form 0/0, so we try L'Hopital's rule again.

$$\lim_{x\to 0}\frac{e^x-1}{e^x+xe^x-1} = \lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x} = \frac{1}{2}$$