Math 2142: Problems on Week 14 material

You do not need to hand in these problems, but they are good practice for the final exam. They cover the material (roughly) from the last week of the semester. Through this assignment, we will use $\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3$ to denote a vector valued position function. Following our notation from class, we let $\mathbf{v}(t) = \mathbf{r}'(t)$ be the vector valued velocity function, $v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$ be the scalar valued speed function and $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ be the vector valued acceleration function.

Problem 1. Let $\mathbf{r}(t) = \langle 3t - t^3, 3t^2, 3t + t^3 \rangle$. Calculate the following quantities at $t = 1$:

$$\mathbf{v}(1), \quad v(t), \quad \mathbf{T}(1), \quad \mathbf{a}(1), \quad \text{and} \quad \mathbf{N}(1)$$

Give the equation (in both parametric form and normal form) for the osculating plane at $t = 1$ and give the decomposition of $\mathbf{a}(1)$ into $\mathbf{T}(1)$ and $\mathbf{N}(1)$.

Problem 2. Let $\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle$ where $a, b > 0$ are constant scalars. Think of $\mathbf{r}(t)$ as the position function of a particle at time t.

- What does the graph of $\mathbf{r}(t)$ look like?
- Calculate the velocity $\mathbf{v}(t)$, the speed $\|\mathbf{v}(t)\|$ and the acceleration $\mathbf{a}(t)$ of the particle. Prove that
$$\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{a}{a^2 + b^2}$$
- Calculate the unit tangent vector $\mathbf{T}(t)$ and the unit normal vector $\mathbf{N}(t)$. Verify that these vectors are orthogonal.
- Find (the normal form for) the equation for the osculating plane when $t = \pi$.
- Find the tangential and normal components of the acceleration.

Problem 3. Let $\mathbf{r}(t)$ be the position function for a particle with velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$. Prove that

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|^2$$

Problem 4. Let $\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3$ denote the position of a particle at time t. Assume that $\mathbf{v}(t)$ is not parallel to $\mathbf{r}(t)$ but that the acceleration function satisfies $\mathbf{a}(t) = -k\,\mathbf{r}(t)$ for some constant $k > 0$.

4(a). Explain why $\mathbf{r}(t) \times \mathbf{v}(t) \neq \overline{0}$ and $\mathbf{r}(t) \times \mathbf{a}(t) = \mathbf{0}$.

4(b). Prove that $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ for some constant vector $\mathbf{c} \neq \mathbf{0}$. (Hint: Use the formula above for the derivative of a cross product to calculate $(\overline{r}(t) \times \overline{v}(t))'$.)

4(c). Explain why the condition $\mathbf{r}(t) \times \mathbf{v}(t) = \mathbf{c}$ in 4(b) implies that the position of the particle is in a plane. (That is, describe the plane that the particle is moving in.)

Problem 5. Find the arc length of the curve $\overline{r}(t) = \langle \sqrt{2}t, e^t, e^{-t} \rangle$ for $0 \leq t \leq 1$.

Problem 6. Reparameterize

$$\overline{r}(t) = \left\langle \frac{2}{t^2+1} - 1, \frac{2t}{t^2+1} \right\rangle$$

with respect to arc length measure from the point $(1, 0)$. Find the point which is distance 2 along this curve from $(1, 0)$.

Problem 7(a). Let $\mathbf{r}(t) : \mathbb{R} \to \mathbb{R}^3$. Prove that

$$\mathbf{r}''(t) = \mathbf{a}(t) = a_T(t)\,\mathbf{T}(t) + a_N(t)\,\mathbf{N}(t)$$

where $a_T(t) = \|\mathbf{r}'(t)\|'$ and $a_N(t) = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| / \|\mathbf{r}'(t)\|$.
(There is not much to show here, just connect the right formulas we have presented in class and everything will work out. The point is that these formulas give an easier method to actually decompose $\mathbf{a}(t)$ into its two components.)

7(b). Use the formula in 7(a) to find $a_T(t)$ and $a_N(t)$ when $\mathbf{r}(t) = \langle t^2, t^2, t^3 \rangle$.

7(c). Calculate $\kappa(t)$ for the curve in 7(b).

Problem 8(a). Consider a curve $\mathcal{C}$ in $\mathbb{R}^2$ given by an equation $y = f(x)$. Prove that the curvature of $\mathcal{C}$ is given by the formula

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Hint: View $\mathcal{C}$ as sitting in the x-y plane in $\mathbb{R}^3$ and parameterize it by

$$\mathbf{r}(x) = \langle x, f(x), 0 \rangle$$

Now you can use our curvature formula in $\mathbb{R}^3$ to calculate the curvature.

8(b). Use the formula in 8(a) to calculate $\kappa(x)$ for $y = x^2$. What happens to $\kappa(x)$ as $x \to \infty$?

Exercises from textbook for practice.

- 14.9: 1-6 (I didn't check how bad the computations are)