Math 2142: Problems on Week 13 material

You do not need to hand in these problems, but they are good practice for the final exam. They cover the material (roughly) from the 13th week of the semester.

Problem 1. Find a unit vector orthogonal to both $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$.

Problem 2. Give the vector form and the parametric form for the following lines in $\mathbb{R}^3$.

- The line through the point $(5,0,1)$ with direction vector $\langle 1,6,-3/2\rangle$.
- The line through the points $(2,4,-3)$ and $(3,-1,1)$.
- The line through $(2,1,0)$ with direction vector orthogonal to $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

Problem 3. Write the vector, parametric and normal forms for the following planes in $\mathbb{R}^3$.

- The plane through the point $(3,4,-1)$ with direction vectors $\langle 0,1,4\rangle$ and $\langle -2,1,6\rangle$.
- The plane through the point $(4,0,-3)$ with normal vector $\mathbf{j}+2\mathbf{k}$.
- The plane passing through the points $(3,-1,2)$, $(8,2,4)$ and $(-1,-2,-3)$.
- The plane through $(-2,8,10)$ and orthogonal to the line $f(t)=\langle 1,0,4\rangle+t\langle 1,2,-3\rangle$.
- The plane through $(6,0,2)$ containing the line $f(t)=\langle 4,3,7\rangle+t\langle -2,5,4\rangle$.

Problem 4. We define the *angle between two planes* to be the angle between their normal vectors. (The planes are parallel if this angle is 0, or equivalently if the normal vectors are multiples of each other.) What is the angle between the planes $x+y+z=1$ and $x-2y+3z=1$?

Hint. Remember that for a plane given by the equation $ax+by+cz=d$, the normal vector to the plane is $\langle a,b,c\rangle$.

Problem 5. Write an equation for the line where the planes $x+y+z=1$ and $x-2y+3z=1$ intersect.

Hint. You need to find a point that lies on both planes. That is, you need to find a common solution for both equations. Also, you need to find the direction for the line. To do this, notice that if the line is in both planes, then it must be perpendicular to the normal vectors for each plane.

Problem 6. Let $F(t)$ be given by

$$F(t) = \frac{2t}{1+t^2}\,\mathbf{i} + \frac{1-t^2}{1+t^2}\,\mathbf{j} + \mathbf{k}$$

Prove that $F(t)$ and $F'(t)$ are orthogonal for every value of t.

Problem 7. Let $F(t) = \langle t, t^2, t^3 \rangle$, $G(t) = \langle e^t, \sqrt{t^2+1}, 3 \rangle$ and $u(t) = 3t^4$.

- Calculate $F'(-2)$, $G'(0)$ and $\int_0^2 F(t)\,dt$.
- Give formulas for the following functions and indicate whether they are vector valued functions or scalar valued functions.

$$-2F \qquad F \cdot G \qquad uF \qquad F \times G \qquad F \circ u = F(u(t)) \qquad \|G\|$$

- Write the equation for the tangent line to $F(t)$ at $t = 3$ and the equation for the tangent line to $G(t)$ at $t = 0$.

Problem 8. Let $F(t)$ and $G(t)$ be differentiable vector valued functions and let $u(t)$ be a differentiable real valued (scalar) function. Let $\overline{c}$ be a constant vector. Prove the following formulas hold. (To make the notation easier, you can assume $F, G : \mathbb{R} \to \mathbb{R}^3$.)

$$(F \cdot G)' = F' \cdot G + F \cdot G'$$
$$(\overline{c} \cdot F)' = \overline{c} \cdot F'(t)$$
$$(uF)' = u'(t)F(t) + u(t)F'(t)$$
$$(F \times G)' = F' \times G + F \times G'$$
$$(F(u(t)))' = u'(t)\,F'(u(t))$$
$$\int_a^b F(t) + G(t)\,dt = \int_a^b F(t)\,dt + \int_a^b G(t)\,dt$$
$$\int_a^b rF(t)\,dt = r\int_a^b F(t)\,dt$$
$$\int_a^b \overline{c} \cdot F(t)\,dt = \overline{c} \cdot \int_a^b F(t)\,dt$$

Problem 9(a). Let $f(x)$ be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ and let $\mathcal{C}$ denote the curve which is the graph of f. Consider the vector valued function $F(t) = \langle t, f(t) \rangle$. Explain why $F(t)$ is a parameterization of $\mathcal{C}$. Which direction along $\mathcal{C}$ does a particle with position $F(t)$ move? Explain why $F(t)$ is a smooth parameterization. ($F(t)$ is a smooth parameterization if $F'(t) \neq \overline{0}$ for all t.)

9(b). Use $f(x)$ to give the usual Single Variable Calculus equation for the tangent line to $f(x)$ at $x = a$. Then use $F(t)$ to give the parametric form of the tangent line to $F(t)$ at $t = a$. Use some algebraic manipulations to show that these tangent lines are the same.

Problem 10. Let $\overline{v} \in \mathbb{R}^3$. Show that $\overline{v} \times \overline{v} = \overline{0}$. (There are lots of ways you might go about this problem. See if you can think of more than one way to solve it.)

Problem 11. Let F be a vector valued function with at least two derivatives and let $G = F \times F'$. Prove that $G' = F \times F''$.

Hint. You may find Problem 10 helpful.

Problem 12. Let F be a vector valued functions with at least three derivatives and let $G = F \cdot F' \times F''$. First, put parentheses in the expression for G so that it makes sense (i.e. do you need to take the dot product or the cross product first?). Second, prove that $G' = F \cdot F' \times F'''$.

Problem 13. Let F be a differentiable vector valued function such that $F'(t) = \overline{0}$ for all t. Explain why there is a constant vector $\overline{C}$ such that $F(t) = \overline{C}$ for all t.

Exercises from textbook for practice. (These are calculation problems with vector valued functions.)

- 3.14: 1, 3
- 14.4: 1, 2, 4, 7, 8, 10, 14
- 14.7: 1-6