Because we are running short on time this semester, I decided to write a set of notes to finish the proof we started in class and to illustrate it with an example. Here is the theorem we started to prove.

Theorem 0.1. *Let $S = \{\overline{u}_1, \ldots, \overline{u}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Every set of $k+1$ many vectors in Span(S) is linearly dependent. Therefore, any set T of vectors in Span(S) with $|T| > k$ is linearly dependent.*

Before giving the proof, I want to do an example that illustrates the method. Consider the following example in $\mathbb{R}^3$. Let

$$S = \{\langle 1, -2, 3\rangle, \langle 5, -13, -3\rangle, \langle -3, 8, 1\rangle\}.$$

You can check that S is linearly independent. Since S is linearly independent and S contains 3 vectors, we know S is a basis for $\mathbb{R}^3$ and hence $\text{Span}(S) = \mathbb{R}^3$. Consider the following set of four vectors in Span(S).

$$T = \{\langle -10, 19, 7\rangle, \langle 4, 9, 4\rangle, \langle 1, -1, 11\rangle, \langle 7, 18, -5\rangle\}.$$

To illustrate the theorem, we need to show that T is linearly dependent. I'm going to do this in a way that parallels the proof we will give below. Let $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_3$ denote the vectors in S in the order given, and let $\overline{v}_1$, $\overline{v}_2$, $\overline{v}_3$ and $\overline{v}_4$ denote the vectors in T in the order given.

Step 1. Since $T \subseteq \text{Span}(S)$, we can write each vector in T as a linear combination of the vectors in S. In practice, this means solving the appropriate systems of linear equations. I won't write out the calculations, but here are the linear combinations.

$$\begin{aligned}
\overline{v}_1 = \langle -10, 19, 7\rangle &= \langle 1, -2, 3\rangle - \langle 5, -13, -3\rangle + 2\langle -3, 8, 1\rangle \\
\overline{v}_2 = \langle 4, 9, 4\rangle &= 2\langle 1, -2, 3\rangle + \langle 5, -13, -3\rangle + \langle -3, 8, 1\rangle \\
\overline{v}_3 = \langle 1, -1, 11\rangle &= 3\langle 1, -2, 3\rangle - \langle 5, -13, -3\rangle - \langle -3, 8, 1\rangle \\
\overline{v}_4 = \langle 7, 18, -5\rangle &= 0\langle 1, -2, 3\rangle + 2\langle 5, -13, -3\rangle + \langle -3, 8, 1\rangle
\end{aligned}$$

Step 2. We want to get rid of the vector $\langle 1, -2, 3\rangle$ on the right side of these equations. We use multiples of the vector $\overline{v}_1$ to do this. For this step, it is important that the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_1$ is not zero.

Since the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_1$ is 1 and the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_2$ is 2, if we take $2\overline{v}_1 - \overline{v}_2$, the copies of $\langle 1, -2, 3\rangle$ will cancel out.

$$2\overline{v}_1 - \overline{v}_2 = -4\langle 5, -13, -3\rangle - 3\langle -3, 8, 1\rangle$$

Similarly, since the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_1$ is 1 and the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_3$ is 3, if we take $3\overline{v}_1 - \overline{v}_3$, the copies of $\langle 1, -2, 3\rangle$ will cancel out.

$$3\overline{v}_1 - \overline{v}_3 = -2\langle 5, -13, -3\rangle + 7\langle -3, 8, 1\rangle$$

Finally, since the coefficient of $\langle 1, -2, 3\rangle$ in $\overline{v}_4$ is already 0, we could just leave $\overline{v}_4$ alone. However, to follow a uniform pattern, we will take the vector $0\overline{v}_1 - \overline{v}_4$, which of course has no copies of $\langle 1, -2, 3\rangle$.

$$0\overline{v}_1 - \overline{v}_4 = -2\langle 5, -13, -3\rangle - \langle -3, 8, 1\rangle$$

Step 3. At this point in the proof, we will apply the induction hypothesis. That is, we now have gotten rid of the vector $\overline{u}_1 = \langle 1, -2, 3\rangle$. If we let

$$\widehat{S} = \{\langle 5, -13, -3\rangle, \langle -3, 8, 1\rangle\}$$
$$\widehat{T} = \{2\overline{v}_1 - \overline{v}_2, 3\overline{v}_1 - \overline{v}_3, 0\overline{v}_1 - \overline{v}_4\}$$

then our equations above show that $\widehat{T} \subseteq \text{Span}(\widehat{S})$. Furthermore, $|\widehat{S}| < |S|$ and $|\widehat{T}| = |\widehat{S}| + 1$, so the induction hypothesis applies to $\widehat{S}$ and $\widehat{T}$. Rather than apply the inductive hypothesis in our example, let's repeat Steps 1 and 2 with $\widehat{S}$ and $\widehat{T}$, which is what induction really does anyway.

Repeated Step 1. We need to write the vectors in $\widehat{T}$ as linear combinations of the vectors in $\widehat{S}$. However, we've already done this above, so we can just copy down those results.

$$\begin{aligned} 2\overline{v}_1 - \overline{v}_2 &= -4\langle 5, -13, -3\rangle - 3\langle -3, 8, 1\rangle \\ 3\overline{v}_1 - \overline{v}_3 &= -2\langle 5, -13, -3\rangle + 7\langle -3, 8, 1\rangle \\ 0\overline{v}_1 - \overline{v}_4 &= -2\langle 5, -13, -3\rangle - \langle -3, 8, 1\rangle \end{aligned}$$

Repeated Step 2. Next, we want to get rid of the vector $\langle 5, -13, -3\rangle$ on the right side of the equations above. We use multiples of the top equation to do this. To make it easy to cancel, I'll multiply the top equation by $-1/2$ which gives us the equations

$$\begin{aligned} -\overline{v}_1 + \frac{1}{2}\overline{v}_2 &= 2\langle 5, -13, -3\rangle + \frac{3}{2}\langle -3, 8, 1\rangle \\ 3\overline{v}_1 - \overline{v}_3 &= -2\langle 5, -13, -3\rangle + 7\langle -3, 8, 1\rangle \\ 0\overline{v}_1 - \overline{v}_4 &= -2\langle 5, -13, -3\rangle - \langle -3, 8, 1\rangle \end{aligned}$$

Adding the top two equations gives

$$2\overline{v}_1 + \frac{1}{2}\overline{v}_2 - \overline{v}_3 = \frac{17}{2}\langle -3, 8, 1\rangle$$

Adding the top and bottom equations above gives

$$-\overline{v}_1 + \frac{1}{2}\overline{v}_2 - \overline{v}_4 = \frac{1}{2}\langle -3, 8, 1\rangle$$

Repeated Step 3. Again, we are back around to the point in the proof when we would apply the induction hypothesis. However, at this point, we can see how to finish the calculation. We have

$$\begin{aligned} 2\overline{v}_1 + \frac{1}{2}\overline{v}_2 - \overline{v}_3 &= \frac{17}{2}\langle -3, 8, 1\rangle \\ -\overline{v}_1 + \frac{1}{2}\overline{v}_2 - \overline{v}_4 &= \frac{1}{2}\langle -3, 8, 1\rangle \end{aligned}$$

Multiplying the top equation by 2 (to get rid of the fractions) and the bottom equation by $-17 \cdot 2$ (to get rid of the fractions and let us cancel the vector $\langle -3, 8, 1\rangle$), we have

$$4\overline{v}_1 + \overline{v}_2 - 2\overline{v}_3 = 17\langle -3, 8, 1\rangle$$
$$34\overline{v}_1 - 17\overline{v}_2 + 34\overline{v}_4 = -17\langle -3, 8, 1\rangle$$

Adding these equations together gives us

$$38\overline{v}_1 - 16\overline{v}_2 - 2\overline{v}_3 + 34\overline{v}_4 = \langle 0, 0, 0\rangle$$

This equation shows that $T = \{\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4\}$ is linearly dependent, which was the goal of this example.

Now, we will give the proof of the theorem above. As you read through it, look back at the example above if it helps to have a more concrete sense of what is going on. The proof proceeds by induction on k.

Base case. For the base case, we have $k = 1$. So, we start with $S = \{\overline{u}_1\}$ is a linearly independent set. For S to be linearly independent means that $\overline{u}_1 \neq \overline{0}$. Let $T = \{\overline{v}_1, \overline{v}_2\} \subseteq \text{Span}(S)$. We need to show that T is linearly dependent. Note that if $\overline{v}_1 = \overline{0}$ or $\overline{v}_2 = \overline{0}$, then T is linearly dependent and we would be done. So, we can assume that $\overline{v}_1 \neq \overline{0}$ and $\overline{v}_2 \neq \overline{0}$.

Since $S = \{\overline{u}_1\}$, $\text{Span}(S) = \{r\overline{u}_1 : r \in \mathbb{R}\}$. Therefore, $T \subseteq \text{Span}(S)$ means that $\overline{v}_1 = r_1\overline{u}_1$ and $\overline{v}_2 = r_2\overline{u}_1$ for some $r_1, r_2 \in \mathbb{R}$. Since $\overline{v}_1 \neq \overline{0}$, we know that $r_1 \neq 0$, and since $\overline{v}_2 \neq \overline{0}$, we know that $r_2 \neq 0$. To see that $T = \{\overline{v}_1, \overline{v}_2\} = \{r_1\overline{u}_1, r_2\overline{u}_2\}$ is linearly dependent, note that

$$r_2\overline{v}_1 - r_1\overline{v}_2 = r_2(r_1\overline{u}_1) - r_1(r_2\overline{u}_1) = r_1r_2\overline{u}_1 - r_1r_2\overline{u}_1 = \overline{0}$$

Since $r_1 \neq 0$, this linear combination has at least one nonzero coefficient, so T is linearly dependent.

Induction case. To make the notation easier, I am going to assume the theorem is true if the set S has $k-1$ many vectors, and prove it remains true if the set S has k many vectors. We start with a linearly independent set $S = \{\overline{u}_1, \ldots, \overline{u}_k\}$ and a set $T \subseteq \text{Span}(S)$ with $k+1$ many vectors

$$T = \{\overline{v}_1, \ldots, \overline{v}_{k+1}\} \subseteq \text{Span}(S)$$

Since each $\overline{v}_i$ vector is in $\text{Span}(S)$, we can write each $\overline{v}_i$ as a linear combination of the vectors in S. This is the analog of Step 1 in the example above, except in our current case, we don't have concrete vectors. So, we use $a_{i,j}$ to stand for the coefficient of $\overline{u}_j$ when we write $\overline{v}_i$ as a linear combination of the vectors in S. That is, we have

$$\begin{array}{rcl} \overline{v}_1 & = & a_{1,1}\overline{u}_1 + a_{1,2}\overline{u}_2 + \cdots + a_{1,k}\overline{u}_k \\ \overline{v}_2 & = & a_{2,1}\overline{u}_1 + a_{2,2}\overline{u}_2 + \cdots + a_{2,k}\overline{u}_k \\ \vdots & \vdots & \vdots \\ \overline{v}_{k+1} & = & a_{k+1,1}\overline{u}_1 + a_{k+1,2}\overline{u}_2 + \cdots + a_{k+1,k}\overline{u}_k \end{array}$$

At this point, we are going to split into two cases. The first case is when the coefficient of $\overline{u}_1$ is 0 in every one of these linear combinations. That is, when each scalar $a_{i,1} = 0$. In this case, the equations above become

$$\begin{aligned}
\overline{v}_1 &= 0\overline{u}_1 + a_{1,2}\overline{u}_2 + \cdots + a_{1,k}\overline{u}_k \\
\overline{v}_2 &= 0\overline{u}_1 + a_{2,2}\overline{u}_2 + \cdots + a_{2,k}\overline{u}_k \\
\vdots & \vdots \vdots \\
\overline{v}_{k+1} &= 0\overline{u}_1 + a_{k+1,2}\overline{u}_2 + \cdots + a_{k+1,k}\overline{u}_k
\end{aligned}$$

or in other words

$$\begin{aligned}
\overline{v}_1 &= a_{1,2}\overline{u}_2 + \cdots + a_{1,k}\overline{u}_k \\
\overline{v}_2 &= a_{2,2}\overline{u}_2 + \cdots + a_{2,k}\overline{u}_k \\
\vdots & \vdots \vdots \\
\overline{v}_{k+1} &= a_{k+1,2}\overline{u}_2 + \cdots + a_{k+1,k}\overline{u}_k
\end{aligned}$$

These equations show that each vector $\overline{v}_i$ is in the span of $\{\overline{u}_2, \ldots, \overline{u}_k\}$. That is, $T \subseteq \text{Span}(\widehat{S})$ where $\widehat{S} = \{\overline{u}_2, \ldots, \overline{u}_k\}$. Since $\widehat{S}$ only contains $k-1$ many vectors, we can apply the induction hypothesis. (That is, we have assumed that the theorem is true for $k - 1$.) The induction hypothesis says that if $T \subseteq \text{Span}(\widehat{S})$ and T has strictly more vectors that $\widehat{S}$, then T is linearly dependent. But, T has $k + 1$ many vectors and $k + 1 > k - 1$, so T is linearly dependent. This completes the first case.

The second case is when at least one of the coefficients of $\overline{u}_1$ is not equal to 0. By renumbering the $\overline{v}_i$ vectors if necessary, we can assume that the coefficient of $\overline{u}_1$ in $\overline{v}_1$ is not zero. That is, $a_{1,1} \neq 0$. We want to go back to our system of equations

$$\begin{aligned}
\overline{v}_1 &= a_{1,1}\overline{u}_1 + a_{1,2}\overline{u}_2 + \cdots + a_{1,k}\overline{u}_k \\
\overline{v}_2 &= a_{2,1}\overline{u}_1 + a_{2,2}\overline{u}_2 + \cdots + a_{2,k}\overline{u}_k \\
\vdots & \vdots \vdots \\
\overline{v}_{k+1} &= a_{k+1,1}\overline{u}_1 + a_{k+1,2}\overline{u}_2 + \cdots + a_{k+1,k}\overline{u}_k
\end{aligned}$$

and try to remove the first vector on the right side of each of these equations as in the example above. For each $\overline{v}_i$ with $i > 1$, let $c_i = a_{i,1}/a_{1,1}$. Note that we are allowed the divide by $a_{1,1}$ because $a_{1,1} \neq 0$. Also, note that

$$c_i\overline{v}_1 = c_i a_{1,1}\overline{u}_1 + c_i a_{1,2}\overline{u}_2 + \cdots + c_i a_{1,k}\overline{u}_k$$

and so

$$c_i\overline{v}_1 = \frac{a_{i,1}}{a_{1,1}} a_{1,1}\overline{u}_1 + c_i a_{1,2}\overline{u}_2 + \cdots + c_i a_{1,k}\overline{u}_k$$

and so

$$c_i\overline{v}_1 = a_{i,1}\overline{u}_1 + c_i a_{1,2}\overline{u}_2 + \cdots + c_i a_{1,k}\overline{u}_k$$

Therefore, if we subtract $c_i\overline{v}_1 - \overline{v}_i$, the $\overline{u}_1$ vectors cancel and we get

$$c_i\overline{v}_1 - \overline{v}_i = (c_i a_{1,2} - a_{i,2})\overline{u}_2 + \cdots + (c_i a_{1,k} - a_{i,k})\overline{u}_k$$

Let $\widehat{T} = \{c_2\overline{v}_1 - \overline{v}_2, c_3\overline{v}_1 - \overline{v}_3, \ldots, c_{k+1}\overline{v}_1 - \overline{v}_{k+1}\}$ and $\widehat{S} = \{\overline{u}_2, \ldots, \overline{u}_k\}$. Notice that $\widehat{T}$ contains k many vectors and that by the equations above, each of these vectors is in Span($\widehat{S}$). Since $\widehat{S}$ contains only $k-1$ vectors, we can apply the inductive hypothesis to $\widehat{S}$ and $\widehat{T}$ to conclude that $\widehat{T}$ is linearly dependent. Therefore, there are scalars $d_2, \ldots, d_{k+1}$ such that at least one of these scalars is not 0 and

$$d_2(c_2\overline{v}_1 - \overline{v}_2) + d_3(c_3\overline{v}_1 - \overline{v}_3) + \cdots + d_{k+1}(c_{k+1}\overline{v}_1 - \overline{v}_{k+1}) = \overline{0}$$

Rearranging these terms, we get

$$(d_2c_2 + d_3c_3 + \cdots + d_{k+1}c_{k+1})\overline{v}_1 - d_2\overline{v}_2 - \cdots - d_{k+1}\overline{v}_{k+1} = \overline{0}$$

Since at least one of $d_2, \ldots, d_{k+1}$ is not 0, this equation shows that T is linearly dependent.