Math 2142 Homework 7 Part 2 Solutions

Problem 1. For each of the following power series, determine for which values of x the series converges. With the Ratio or Root Tests, you often find an open interval of the form $(-r, r)$ on which the series converges. Be sure to check whether the series converges or diverges at the endpoints of this interval.

1(a).

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n}$$

Solution. Applying the Ratio Test,

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+2)2^{n+1}} \cdot \frac{(n+1)2^n}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{2} \cdot \frac{n+1}{n+2} = \frac{|x|}{2}$$

Setting $|x|/2 < 1$, we see that we get (absolute) convergence when $|x| < 2$ and so the radius of convergence is 2. To find the interval of convergence, we need to test the endpoints $x = 2$ and $x = -2$. For $x = 2$, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{1}{n+1}$$

which diverges by the Limit Comparison Test with the series $\sum 1/n$. For the other endpoint $x = -2$, we get

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

which converges by the Alternating Series Test. Therefore, the interval of convergence is $[-2, 2)$.

1(b).

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt[4]{n}}$$

Solution. Applying the Ratio Test,

$$\lim_{n\to\infty} \frac{2^{n+1}|x|^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{2^n|x|^n} = \lim_{n\to\infty} 2|x|\sqrt[4]{\frac{n}{n+1}} = 2|x|\sqrt[4]{1} = 2|x|$$

Setting $2|x| < 1$, we see that the series converges (absolutely) for $|x| < 1/2$ and so the radius of convergence is $1/2$. To test the endpoints, we look at $x = 1/2$ and $x = -1/2$. First, for $x = 1/2$ we have

$$\sum_{n=1}^{\infty} \frac{(-2)^n(1/2)^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$$

which converges by the Alternating Series Test. Second, for $x = -1/2$, we have

$$\sum_{n=1}^{\infty} \frac{(-2)^n(-1/2)^n}{\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$$

which diverges since it is a p-series with $p = 1/4$. Therefore, the interval of convergence is $(-1/2, 1/2]$.

1(c).

$$\sum_{n=0}^{\infty} \sqrt{n} x^n$$

Solution. Applying the Ratio Test, we have

$$\lim_{n\to\infty} \frac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim_{n\to\infty} |x|\sqrt{\frac{n+1}{n}} = |x|\sqrt{1} = |x|$$

Setting $|x| < 1$ we see that the series converges (absolutely) if $|x| < 1$ and the radius of convergence is 1. To check the endpoints, first consider $x = 1$. This give $\sum_{n=0}^{\infty} \sqrt{n}$ which diverges by the Test for Divergence. Second, for $x = -1$, we have $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$ which also diverges by the Test for Divergence. Therefore, the interval of convergence is $(-1, 1)$.

1(d).

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^n}$$

Solution. Applying the Root Test, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{|x-2|^n}{n^n}} = \lim_{n\to\infty} \frac{|x-2|}{n} = 0$$

for all x. Therefore, the series converges absolutely for all x and the radius of convergence is infinite.

1(e).

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x+5)^n}{n5^n}$$

Solution. Applying the Ration Test, we have

$$\lim_{n\to\infty} \frac{|x+5|^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n5^n}{|x+5|^n} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{|x+5|}{5} = \frac{|x+5|}{5}$$

Setting $|x+5|/5 < 1$, we get that the series converges absolutely if $|x+5| < 5$ and so the radius of convergence is 5. Rewriting this inequality, we have $-5 < x + 5 < 5$ which means $-10 < x < 0$. Therefore, the endpoints to check are $x = -10$ and $x = 0$.

For $x = -10$, we have the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{(-5)^n}{n5^n} = \sum_{n=1}^{\infty}(-1)^n \frac{(-1)^n 5^n}{n5^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges. For $x = 0$, we have the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{5^n}{n5^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$$

which converges by the Alternating Series Test. Therefore, the interval of convergence is $(-10, 0]$.

Problem 2. Each of the following series can be rewritten as a geometric series. Use this fact to determine for which values of x the series converges and what the value of the series is for each such x.

2(a).

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}}$$

Solution. Rewriting our series, we have

$$\sum_{n=0}^{\infty} \frac{x^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1 - x/3} = \frac{1}{3 - x}$$

and we know this equality holds if and only if $|x|/3 < 1$, which is the same as $|x| < 3$.

2(b).

$$\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

Solution. Rewriting our series, we have

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = \frac{1}{1 - (-x^2)} = \frac{1}{1 + x^2}$$

and this equality holds if and only if $|-x^2| < 1$, which is the same as $|x| < 1$.