Math 2142 Homework 6 Solutions

Problem 1. Let r be a fixed real number. Show that e^r is the limit of the sequence given by

$$a_n = \left(1 + \frac{r}{n}\right)^n$$

Solution. We replace n by the real variable x and rewrite the expression as follows.

$$\lim_{x\to\infty} \left(1 + \frac{r}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1+\frac{r}{x}\right)}$$

Since the exponential function is continuous, we can calculate the limit by taking the limit of the term in the exponent. We first rewrite the expression to get it into the form 0/0.

$$\lim_{x\to\infty} x \ln\left(1 + \frac{r}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{r}{x}\right)}{\frac{1}{x}}$$

Next, we apply L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{\ln\left(1 + \frac{r}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\left(1 + \frac{r}{x}\right)^{-1} \frac{-r}{x^2}}{\frac{-1}{x^2}} = \lim_{x\to\infty} \frac{r}{\left(1 + \frac{r}{x}\right)} = r$$

Therefore, the limit of the sequence is e^r.

Problem 2. Do the following problems in Exercises 10.4 on page 382: 1, 2, 4, 6, 7.

Solution to Exercise 1. Replacing n by the real variable x, we calculate

$$\lim_{x\to\infty} \frac{x}{x+1} - \frac{x+1}{x} = 1 - 1 = 0$$

where in the second step we use L'Hospital's Rule on each of the fractions.

Solution to Exercise 2. We replace n by the real variable x, combine the terms into one fraction, do some algebraic simplifications and apply L'Hospital's Rule.

$$\lim_{x\to\infty} \frac{x^2 \cdot x - (x^2+1)(x+1)}{(x+1)\cdot x} = \lim_{x\to\infty} \frac{x^3 - (x^3 + x^2 + x + 1)}{x^2 + x} = \lim_{x\to\infty} \frac{-x^2 - x - 1}{x^2 + x} = -1$$

Solution to Exercise 4. Replacing n by the real variable x, we use L'Hospital's Rule to calculate

$$\lim_{x\to\infty} \frac{x^2 + 3x - 2}{5x^2} = \frac{1}{5}$$

Solution to Exercise 6. The terms in this sequence alternate between 0 and 2, so the sequence diverges.

Solution to Exercise 7. The numerators of the terms in this sequence alternate between 0 and 2. Therefore, for any n, we have

$$0 \leq \frac{1+(-1)^n}{n} \leq \frac{2}{n}$$

Since $\lim_{n\to\infty} 2/n = 0$, we have $\lim_{n\to\infty}(1+(-1)^n)/n = 0$ by the Squeeze Theorem.

Problem 3. Do the following problems from Exercises 10.9 in the textbook: 2, 3, 4, 5.

Solution to Exercise 2. We would like to use the formula for the sum of a geometric series, so we need to shift the indices to start with $n = 0$. Then, we can pull out the constant 2 and apply the geometric series formula.

$$\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = \sum_{n=0}^{\infty} \frac{2}{3^n} = 2 \cdot \sum_{n=0}^{\infty} \frac{1}{3^n} = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = 2 \cdot \frac{1}{1-1/3} = 2 \cdot \frac{1}{2/3} = 2 \cdot \frac{3}{2} = 3$$

Solution to Exercise 3. For this problem, we begin by finding the partial fraction decomposition.

$$\frac{1}{n^2-1} = \frac{1}{(n+1)(n-1)} = \frac{a}{n+1} + \frac{b}{n-1}$$

Multiplying through by $(n+1)(n-1)$ gives us $1 = a(n-1) + b(n+1)$. Plugging in $n = 1$ shows $b = 1/2$ and plugging in $n = -1$ shows $a = -1/2$. Therefore, we have

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n-1} - \frac{1}{n+1}$$

Write out the first few partial sums for the series $\sum_{n=2}^{\infty} 1/(n-1) - 1/(n+1)$ to see how the telescoping cancelation works.

$$s_2 = 1 - \frac{1}{3}$$
$$s_3 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right)$$
$$s_4 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$$

To continue the pattern, remember that in general $s_{n+1} = s_n + a_{n+1}$. So, we can use our simplified version of s_n and just add the next term a_{n+1} to find s_{n+1}.

$$s_5 = \left(1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) = 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$$
$$s_6 = \left(1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) = 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$$

At this point, the general pattern emerges. We have

$$s_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1} = \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$$

(It is a good exercise to prove this formula by induction on $n \geq 2$.) Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2} \cdot \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$$

Solution to Exercise 4. We calculate the two parts of the sum separately. In each case, we need to shift the indices to apply the formula for a geometric series.

$$\sum_{n=1}^{\infty} \frac{2^n}{6^n} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{6^{n+1}} = \sum_{n=0}^{\infty} \frac{2 \cdot 2^n}{6 \cdot 6^n} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{1-1/3} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{6^n} = \sum_{n=0}^{\infty} \frac{3^{n+1}}{6^{n+1}} = \sum_{n=0}^{\infty} \frac{3 \cdot 3^n}{6 \cdot 6^n} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-1/2} = \frac{1}{2} \cdot 2 = 1$$

Therefore, we have

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{6^n} + \sum_{n=1}^{\infty} \frac{3^n}{6^n} = \frac{1}{2} + 1 = \frac{3}{2}$$

Solution to Exercise 5. We begin with some algebra.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n(n+1)}} = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{\sqrt{n(n+1)}} - \frac{\sqrt{n}}{\sqrt{n(n+1)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

Write out the first few partial sums to find the telescoping pattern.

$$s_1 = 1 - \frac{1}{\sqrt{2}}$$
$$s_2 = \left(1 - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}\right) = 1 - \frac{1}{\sqrt{3}}$$
$$s_3 = \left(1 - \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}\right) = 1 - \frac{1}{\sqrt{4}}$$

The general pattern is

$$s_n = 1 - \frac{1}{\sqrt{n+1}}$$

(As above, it is a good exercise to prove this formula by induction on $n \geq 1$.) Using this formula, we have

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = \lim_{n\to\infty} 1 - \frac{1}{\sqrt{n+1}} = 1$$

Problem 4. Do the following problems from Exercises 10.14 in the textbook: 1, 6, 7, 9.

Solution to Exercise 1. Multiplying out, we have

$$\sum_{n=1}^{\infty} \frac{n}{(4n-3)(4n-1)} = \sum_{n=1}^{\infty} \frac{n}{16n^2 - 16n + 3}$$

We use the Limit Comparison Test with the divergent series $\sum_{n=1}^{\infty} 1/n$.

$$\lim_{n\to\infty} \frac{n/(16n^2-16n+3)}{1/n} = \lim_{n\to\infty} \frac{n}{16n^2-16n+3} \cdot n = \lim_{n\to\infty} \frac{n^2}{16n^2-16n+3}$$

Switching from n to a real variable x and applying L'Hospital's Rule, we see that this limit converges to $1/16$. Since $1/16 > 0$, the Limit Comparison Test tells us that our original series diverges.

Solution to Exercise 6. When n is even, we have $2 + (-1)^n = 2 + 1 = 3$ and when n is odd, we have $2 + (-1)^n = 2 - 1 = 1$. Therefore, regardless of the value of n, we have

$$\frac{2 + (-1)^n}{2^n} \leq \frac{3}{2^n}$$

The series $\sum_{n=1}^{\infty} 3/2^n$ converges because it is a geometric series with $r = 1/2$. Therefore, by the Comparison Test, our original series converges as well.

Solution to Exercise 7. We begin by simplifying our series using algebra.

$$\sum_{n=1}^{\infty} \frac{n!}{(n+2)!} = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges (by the p-test with $p = 2$) and

$$\frac{1}{n^2+3n+2} \leq \frac{1}{n^2}$$

we have by the Comparison Test that our original series converges as well. (Notice that you could also use partial fractions to write this series in a telescoping fashion and figure out exactly what it converges to.)

Solution to Exercise 9. We begin with some simplifying algebra:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$$

We use the Limit Comparison Test with the divergent series $\sum_{n=1}^{\infty} 1/n$.

$$\lim_{n\to\infty} \frac{1/n}{1/\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{\sqrt{n^2+n}}{n} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$$

Since $1 > 0$, the Limit Comparison Test tells us that our original series diverges as well.

Problem 5. Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = A$. Prove that for any constant c, the sequence $\{ca_n\}$ has limit $\lim_{n\to\infty} ca_n = cA$. (You need to work with ε and N to do this proof. Remember that c could be negative!)

Solution. To prove that $\lim_{n\to\infty} ca_n = cA$, we need to show that for any $\varepsilon > 0$, there is a natural number N such that for all $n > N$, $|ca_n - cA| < \varepsilon$. Fix a given $\varepsilon > 0$ and we need to find the corresponding N. Note that if $c = 0$, then $|ca_n - cA| = 0 < \varepsilon$, so we would be done. Therefore, I will assume that $c \neq 0$.

(First, I will do the usual side calculation as we did last semester. This calculation is not formally part of the proof, but it tells us how to proceed. We eventually want to show that if $n > N$, then $|ca_n - cA| < \varepsilon$. We rewrite $|ca_n - cA| < \varepsilon$ as $|c| \cdot |a_n - A| < \varepsilon$. From here, we see that we want to pick N so that if $n > N$, then $|a_n - A| < \varepsilon/|c|$. But, we know $\lim_{n\to\infty} a_n = A$, so we can make $|a_n - A| < \varepsilon/|c|$ by working with $\widehat{\varepsilon} = \varepsilon/|c|$ in the definition of $\lim_{n\to\infty} a_n = A$. This ends the side calculation as we now know how to proceed.)

Let $\widehat{\varepsilon} = \varepsilon/|c|$. Because $\lim_{n\to\infty} a_n = A$, we can fix a natural number N such that if $n > N$, then $|a_n - A| < \widehat{\varepsilon}$. We claim that this value of N works for our limit as well. To check that N works, let $n > N$. By the choice of N, we know $|a_n - A| < \widehat{\varepsilon}$. Since $\widehat{\varepsilon} = \varepsilon/|c|$, this means $|a_n - A| < \varepsilon/|c|$. Multiplying by $|c|$ (which is positive), we have $|c| \cdot |a_n - A| < \varepsilon$. Finally, putting the constant c inside the absolute value signs, we have $|ca_n - cA| < \varepsilon$ as required.