# Math 2142 Homework 5 Part 2 Solutions

**Problem 1.** Consider a pair of linear 2nd order differential equations with the same left side.

$$\begin{aligned} y'' + P_1(x)\, y' + P_2(x)\, y &= Q_1(x) \\ y'' + P_1(x)\, y' + P_2(x)\, y &= Q_2(x) \end{aligned}$$

Prove that if $y_1(x)$ is a solution to the top equation and $y_2(x)$ is a solution to the bottom equation, then $y_1(x) + y_2(x)$ is a solution to

$$y'' + P_1(x)\, y' + P_2(x)\, y = Q_1(x) + Q_2(x)$$

**Solution.** Let $L(y) = y'' + P_1(x)y' + P_2(x)y$. From our work in class, we know that $L(y)$ is a linear operator. The assumptions of this problem tell us that $L(y_1) = Q_1(x)$ and $L(y_2) = Q_2(x)$. Using the linearity of $L(y)$, we know $L(y_1 + y_2) = L(y_1) + L(y_2) = Q_1(x) + Q_2(x)$, which means that $y_1 + y_2$ is a solution to $y'' + P_1(x)y' + P_2(x)y = Q_1(x) + Q_2(x)$.

Alternately, you can solve this problem directly. We are given that

$$\begin{aligned} y_1'' + P_1(x)\, y_1' + P_2(x)\, y_1 &= Q_1(x) \\ y_2'' + P_1(x)\, y_2' + P_2(x)\, y_2 &= Q_2(x) \end{aligned}$$

We calculate

$$\begin{aligned} &\frac{d^2}{dx^2}(y_1 + y_2) + P_1(x)\,\frac{d}{dx}(y_1 + y_2) + P_2(x)(y_1 + y_2) \\ &= (y_1'' + y_2'') + P_1(x)(y_1' + y_2') + P_2(x)(y_1 + y_2) \\ &= (y_1'' + P_1(x)y_1' + P_2(x)y_1) + (y_2'' + P_1(x)y_2' + P_2(x)y_2) \\ &= Q_1(x) + Q_2(x) \end{aligned}$$

and hence $y_1 + y_2$ satisfies the required equation.

**Problem 2(a).** Find the general solution for

$$y'' + y = e^x + x^3 + x + 2$$

**2(b).** Find the particular solution to the equation in 2(a) satisfying $y(0) = 2$ and $y'(0) = 0$.

**Solution.** For 5(a), we break the problem into three pieces. For the first piece, we solve the homogeneous equation $y'' + y = 0$. The characteristic polynomial is $r^2 + 1$ which has roots $r = \pm i$. Using $r = i = 0 + i$, a complex solution to the homogeneous equation is

$$y = e^{0x} \cos x + ie^{0x} \sin x = \cos x + i \sin x$$

and therefore the general (real) solution is $y = c_1 \cos x + c_2 \sin x$.

For the second piece, we find a single solution to the non-homogeneous equation $y''+y=e^x$ by guessing $y=Ae^x$. Taking derivatives, we have $y'=y''=Ae^x$. Plugging into the non-homogeneous equation gives

$$\begin{aligned} y''+y &= e^x \\ Ae^x+Ae^x &= e^x \\ 2Ae^x &= e^x \end{aligned}$$

and so $A=1/2$. This tells us that one solution to $y''+y=e^x$ is $y=1/2\,e^x$.

For the third piece, we find a single solution to the non-homogeneous equation $y''+y=x^3+x+2$ by guessing $y=Ax^3+Bx^2+Cx+D$. Taking derivatives gives us

$$\begin{aligned} y' &= 3Ax^2+2Bx+C \\ y'' &= 6Ax+2B \end{aligned}$$

Plugging into the non-homogeneous equation gives us

$$\begin{aligned} y''+y &= x^3+x+2 \\ (6Ax+2B)+(Ax^3+Bx^2+Cx+D) &= x^3+x+2 \\ Ax^3+Bx^2+(6A+C)x+(2B+D) &= x^3+x+2 \end{aligned}$$

Comparing the coefficients on each side of the equation gives $A=1$, $B=0$, $6A+C=1$ and $2B+D=2$. Solving for $C$ and $D$ gives $C=-5$ and $D=2$. Therefore, a single solution to this non-homogeneous equation is $y=x^3-5x+2$.

Putting the three pieces together, the general solution for the original non-homogeneous equation is

$$y=c_1\cos x+c_2\sin x+\frac{1}{2}e^x+x^3-5x+2$$

To find the specific solution for 3(b), we first use $y(0)=2$ to get

$$c_1\cos 0+c_2\sin 0+\frac{1}{2}e^0+0^3-5(0)+2=2$$

which means $c_1+1/2+2=2$ so $c_1=-1/2$. Our specific solution now has the form

$$y=-\frac{1}{2}\cos x+c_2\sin x+\frac{1}{2}e^x+x^3-5x+2$$

Taking a derivative gives us

$$y'=\frac{1}{2}\sin x-c_2\cos x+\frac{1}{2}e^x+3x^2-5$$

Plugging in $y'(0)=0$ we get

$$\frac{1}{2}\sin 0-c_2\cos 0+\frac{1}{2}e^0+3(0^2)-5=0$$

which means $c_2 = 9/2$. Therefore, the specific solution is

$$y = -\frac{1}{2}\cos x + \frac{9}{2}\sin x + \frac{1}{2}e^x + x^3 - 5x + 2$$

**Problem 3.** Use Laplace transforms to solve the following initial value problems.

$$\begin{gathered} y'' - y' - 6y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = -1 \\ y'' - 2y' + 2y = 0 \text{ with } y(0) = 0 \text{ and } y'(0) = 1 \\ y'' + 9y = 1 \text{ with } y(0) = y'(0) = 0 \\ y'' + 4y = \sin 3t \text{ with } y(0) = y'(0) = 0 \end{gathered}$$

**Solution.** Applying the Laplace transform to the first equation

$$y'' - y' - 6y = 0 \text{ with } y(0) = 1 \text{ and } y'(0) = -1$$

gives

$$\begin{aligned} \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} &= 0 \\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) - (s\mathcal{L}\{y\} - y(0)) - 6\mathcal{L}\{y\} &= 0 \\ \mathcal{L}\{y\}(s^2 - s - 6) - s + 1 + 1 &= 0 \\ \mathcal{L}\{y\}(s-3)(s+2) &= s - 2 \\ \mathcal{L}\{y\} &= \frac{s-2}{(s-3)(s+2)} \end{aligned}$$

We decompose the righthand side using partial fractions:

$$\frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

Multiplying through by $(s-3)(s+2)$ leaves us with $s - 2 = A(s+2) + B(s-3)$. Plugging in $s = 3$ gives $1 = A(5)$, so $A = 1/5$. Plugging in $s = -2$ gives $-4 = B(-5)$, so $B = 4/5$. Therefore, we have

$$\mathcal{L}\{y\} = \frac{s-2}{(s-3)(s+2)} = \frac{1}{5} \cdot \frac{1}{s-3} + \frac{4}{5} \cdot \frac{1}{s+2}$$

To invert the Laplace transform, notice that $\mathcal{L}\{e^{3t}\} = 1/(s-3)$ and $\mathcal{L}\{e^{-2t}\} = 1/(s+2)$. Therefore, our solution is

$$y = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

Applying the Laplace transform to the second equation

$$y'' - 2y' + 2y = 0 \text{ with } y(0) = 0 \text{ and } y'(0) = 1$$

gives

$$\begin{aligned}\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= 0\\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 2(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} &= 0\\ \mathcal{L}\{y\}(s^2 - 2s + 2) - 1 &= 0\\ \mathcal{L}\{y\} &= \frac{1}{s^2 - 2s + 2}\end{aligned}$$

The polynomial $s^2 - 2s + 2$ does not factor, so the righthand side of the last equation is already in a partial fraction form. To invert the Laplace transform, we need to complete the square: $s^2 - 2s + 2 = (s-1)^2 + 1$. Therefore,

$$\mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s-1)^2 + 1}$$

Since $\mathcal{L}\{e^t \sin t\} = 1/((s-1)^2 + 1)$, our solution is

$$y = e^t \sin t$$

Applying the Laplace transform to the third equation

$$y'' + 9y = 1 \text{ with } y(0) = y'(0) = 0$$

gives

$$\begin{aligned}\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} &= \mathcal{L}\{1\}\\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} &= \frac{1}{s}\\ \mathcal{L}\{y\}(s^2 + 9) &= \frac{1}{s}\\ \mathcal{L}\{y\} &= \frac{1}{s(s^2 + 9)}\end{aligned}$$

We decompose the righthand side using partial fractions:

$$\frac{1}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}$$

Multiplying through by $s(s^2 + 9)$ gives $1 = A(s^2 + 9) + (Bs + C)s$. Expanding and collecting terms gives $1 = (A + B)s^2 + Cs + 9A$. Therefore, $A = 1/9$, $C = 0$ and $B = -1/9$ and we have

$$\mathcal{L}\{y\} = \frac{1}{s(s^2 + 9)} = \frac{1}{9} \cdot \frac{1}{s} - \frac{1}{9} \cdot \frac{s}{s^2 + 9}$$

Since $\mathcal{L}\{1\} = 1/s$ and $\mathcal{L}\{\cos 3t\} = s/(s^2 + 9)$, the solution is

$$y = \frac{1}{9} - \frac{1}{9}\cos 3t$$

Applying the Laplace transform to the last equation

$$y'' + 4y = \sin 3t \text{ with } y(0) = y'(0) = 0$$

gives

$$\begin{aligned}
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} &= \mathcal{L}\{\sin 3t\} \\
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} &= \frac{3}{s^2+9} \\
\mathcal{L}\{y\}(s^2+4) &= \frac{3}{s^2+9} \\
\mathcal{L}\{y\} &= \frac{3}{(s^2+9)(s^2+4)}
\end{aligned}$$

We decompose the righthand side using partial fractions

$$\frac{3}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$

Multiplying through by $(s^2+9)(s^2+4)$ gives $3 = (As+B)(s^2+4) + (Cs+D)(s^2+9)$. Expanding and collecting similar terms gives

$$3 = (A+C)s^3 + (B+D)s^2 + (4A+9C)s + (4B+9D)$$

Comparing coefficients on each side of the equation, we have two linear systems to solve. First, we see that $A + C = 0$ and $4A + 9C = 0$. The first of these equations gives $A = -C$, which when plugged into the second equation, gives $-4C + 9C = 0$, so $C = 0$ and hence $A = 0$.

Second, we also have $B + D = 0$ and $4B + 9D = 3$. The first of these equations gives $B = -D$, which when plugged into the second equation, gives $-4D + 9D = 3$, which means $D = 3/5$. Since $B = -D$, this means $B = -3/5$. Therefore, we have

$$\mathcal{L}\{y\} = \frac{3}{(s^2+9)(s^2+4)} = -\frac{3}{5}\cdot\frac{1}{s^2+9} + \frac{3}{5}\cdot\frac{1}{s^2+4}$$

To invert the Laplace transform, note that $\mathcal{L}\{\sin 3t\} = 3/(s^2+9)$ and $\mathcal{L}\{\sin 2t\} = 2/(s^2+4)$. Therefore,

$$\mathcal{L}\{y\} = -\frac{3}{5}\cdot\frac{1}{3}\cdot\frac{3}{s^2+9} + \frac{3}{5}\cdot\frac{1}{2}\cdot\frac{2}{s^2+4} = -\frac{1}{5}\cdot\frac{3}{s^2+9} + \frac{3}{10}\cdot\frac{2}{s^2+4}$$

and the solution is

$$y = -\frac{1}{5}\sin 3t + \frac{3}{10}\sin 2t$$