Math 2142 Homework 5 Part 2 Solutions

Problem 1. Consider a pair of linear 2nd order differential equations with the same left side.

\[ y'' + P_1(x) y' + P_2(x) y = Q_1(x) \]
\[ y'' + P_1(x) y' + P_2(x) y = Q_2(x) \]

Prove that if \( y_1(x) \) is a solution to the top equation and \( y_2(x) \) is a solution to the bottom equation, then \( y_1(x) + y_2(x) \) is a solution to

\[ y'' + P_1(x) y' + P_2(x) y = Q_1(x) + Q_2(x) \]

Solution. Let \( L(y) = y'' + P_1(x) y' + P_2(x)y \). From our work in class, we know that \( L(y) \) is a linear operator. The assumptions of this problem tell us that \( L(y_1) = Q_1(x) \) and \( L(y_2) = Q_2(x) \). Using the linearity of \( L(y) \), we know \( L(y_1 + y_2) = L(y_1) + L(y_2) = Q_1(x) + Q_2(x) \), which means that \( y_1 + y_2 \) is a solution to \( y'' + P_1(x) y' + P_2(x)y = Q_1(x) + Q_2(x) \).

Alternately, you can solve this problem directly. We are given that

\[ y_1'' + P_1(x) y_1' + P_2(x) y_1 = Q_1(x) \]
\[ y_2'' + P_1(x) y_2' + P_2(x) y_2 = Q_2(x) \]

We calculate

\[
\frac{d^2}{dx^2} (y_1 + y_2) + P_1(x) \frac{d}{dx} (y_1 + y_2) + P_2(x)(y_1 + y_2) \\
= (y_1'' + y_2'') + P_1(x)(y_1' + y_2') + P_2(x)(y_1 + y_2) \\
= (y'' + P_1(x)y' + P_2(x)y) + (y'' + P_1(x)y' + P_2(x)y) \\
= Q_1(x) + Q_2(x)
\]

and hence \( y_1 + y_2 \) satisfies the required equation.

Problem 2(a). Find the general solution for

\[ y'' + y = e^x + x^3 + x + 2 \]

2(b). Find the particular solution to the equation in 2(a) satisfying \( y(0) = 2 \) and \( y'(0) = 0 \).

Solution. For 5(a), we break the problem into three pieces. For the first piece, we solve the homogeneous equation \( y'' + y = 0 \). The characteristic polynomial is \( r^2 + 1 \) which has roots \( r = \pm i \). Using \( r = i = 0 + i \), a complex solution to the homogeneous equation is

\[ y = e^{0x} \cos x + ie^{0x} \sin x = \cos x + i \sin x \]

and therefore the general (real) solution is \( y = c_1 \cos x + c_2 \sin x \).
For the second piece, we find a single solution to the non-homogeneous equation $y'' + y = e^x$ by guessing $y = Ae^x$. Taking derivatives, we have $y' = y'' = Ae^x$. Plugging into the non-homogeneous equation gives

\[
\begin{align*}
    y'' + y &= e^x \\
    Ae^x + Ae^x &= e^x \\
    2Ae^x &= e^x
\end{align*}
\]

and so $A = 1/2$. This tells us that one solution to $y'' + y = e^x$ is $y = 1/2 e^x$.

For the third piece, we find a single solution to the non-homogeneous equation $y'' + y = x^3 + x + 2$ by guessing $y = Ax^3 + Bx^2 + Cx + D$. Taking derivatives gives us

\[
\begin{align*}
    y' &= 3Ax^2 + 2Bx + C \\
    y'' &= 6Ax + 2B
\end{align*}
\]

Plugging into the non-homogeneous equation gives us

\[
\begin{align*}
    y'' + y &= x^3 + x + 2 \\
    (6Ax + 2B) + (Ax^3 + Bx^2 + Cx + D) &= x^3 + x + 2 \\
    Ax^3 + Bx^2 + (6A + C)x + (2B + D) &= x^3 + x + 2
\end{align*}
\]

Comparing the coefficients on each side of the equation gives $A = 1$, $B = 0$, $6A + C = 1$ and $2B + D = 2$. Solving for $C$ and $D$ gives $C = -5$ and $D = 2$. Therefore, a single solution to this non-homogeneous equation is $y = x^3 - 5x + 2$.

Putting the three pieces together, the general solution for the original non-homogeneous equation is

\[
y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^x + x^3 - 5x + 2
\]

To find the specific solution for 3(b), we first use $y(0) = 2$ to get

\[
c_1 \cos 0 + c_2 \sin 0 + \frac{1}{2} e^0 + 0^3 - 5(0) + 2 = 2
\]

which means $c_1 + 1/2 + 2 = 2$ so $c_1 = -1/2$. Our specific solution now has the form

\[
y = -\frac{1}{2} \cos x + c_2 \sin x + \frac{1}{2} e^x + x^3 - 5x + 2
\]

Taking a derivative gives us

\[
y' = \frac{1}{2} \sin x - c_2 \cos x + \frac{1}{2} e^x + 3x^2 - 5
\]

Plugging in $y'(0) = 0$ we get

\[
\frac{1}{2} \sin 0 - c_2 \cos 0 + \frac{1}{2} e^0 + 3(0^2) - 5 = 0
\]
which means \( c_2 = 9/2 \). Therefore, the specific solution is

\[
y = -\frac{1}{2} \cos x + \frac{9}{2} \sin x + \frac{1}{2} e^x + x^3 - 5x + 2
\]

**Problem 3.** Use Laplace transforms to solve the following initial value problems.

\[
y'' - y' - 6y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = -1
\]
\[
y'' - 2y' + 2y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1
\]
\[
y'' + 9y = 1 \quad \text{with} \quad y(0) = y'(0) = 0
\]
\[
y'' + 4y = \sin 3t \quad \text{with} \quad y(0) = y'(0) = 0
\]

**Solution.** Applying the Laplace transform to the first equation

\[
y'' - y' - 6y = 0 \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = -1
\]
gives

\[
\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = 0
\]
\[
s^2\mathcal{L}\{y\} - sy(0) - y'(0) - (s\mathcal{L}\{y\} - y(0)) - 6\mathcal{L}\{y\} = 0
\]
\[
\mathcal{L}\{y\}(s^2 - s - 6) - s + 1 + 1 = 0
\]
\[
\mathcal{L}\{y\} = \frac{s - 2}{(s - 3)(s + 2)}
\]

We decompose the righthand side using partial fractions:

\[
\frac{s - 2}{(s - 3)(s + 2)} = \frac{A}{s - 3} + \frac{B}{s + 2}
\]

Multiplying through by \((s - 3)(s + 2)\) leaves us with \(s - 2 = A(s + 2) + B(s - 3)\). Plugging in \(s = 3\) gives \(1 = A(5)\), so \(A = 1/5\). Plugging in \(s = -2\) gives \(-4 = B(-5)\), so \(B = 4/5\). Therefore, we have

\[
\mathcal{L}\{y\} = \frac{s - 2}{(s - 3)(s + 2)} = \frac{1}{5} \cdot \frac{1}{s - 3} + \frac{4}{5} \cdot \frac{1}{s + 2}
\]

To invert the Laplace transform, notice that \(\mathcal{L}\left\{e^{3t}\right\} = 1/(s - 3)\) and \(\mathcal{L}\left\{e^{-2t}\right\} = 1/(s + 2)\). Therefore, our solution is

\[
y = \frac{1}{5} e^{3t} + \frac{4}{5} e^{-2t}
\]

Applying the Laplace transform to the second equation

\[
y'' - 2y' + 2y = 0 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 1
\]
The polynomial $s^2 - 2s + 2$ does not factor, so the righthand side of the last equation is already in a partial fraction form. To invert the Laplace transform, we need to complete the square: $s^2 - 2s + 2 = (s - 1)^2 + 1$. Therefore,

$$\mathcal{L}\{y\} = \frac{1}{s^2 - 2s + 2} = \frac{1}{(s - 1)^2 + 1}$$

Since $\mathcal{L}\{e^t \sin t\} = 1/((s - 1)^2 + 1)$, our solution is

$$y = e^t \sin t$$

Applying the Laplace transform to the third equation

$$y'' + 9y = 1$$

with $y(0) = y'(0) = 0$ gives

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{1\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 9\mathcal{L}\{y\} = \frac{1}{s}$$

$$\mathcal{L}\{y\}(s^2 + 9) = \frac{1}{s}$$

$$\mathcal{L}\{y\} = \frac{1}{s(s^2 + 9)}$$

We decompose the righthand side using partial fractions:

$$\frac{1}{s(s^2 + 9)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 9}$$

Multiplying through by $s(s^2 + 9)$ gives $1 = A(s^2 + 9) + (Bs + C)s$. Expanding and collecting terms gives $1 = (A + B)s^2 + Cs + 9A$. Therefore, $A = 1/9$, $C = 0$ and $B = -1/9$ and we have

$$\mathcal{L}\{y\} = \frac{1}{s(s^2 + 9)} = \frac{1}{9} \cdot \frac{1}{s} - \frac{1}{9} \cdot \frac{s}{s^2 + 9}$$

Since $\mathcal{L}\{1\} = 1/s$ and $\mathcal{L}\{\cos 3t\} = s/(s^2 + 9)$, the solution is

$$y = \frac{1}{9} - \frac{1}{9} \cos 3t$$
Applying the Laplace transform to the last equation

\[ y'' + 4y = \sin 3t \text{ with } y(0) = y'(0) = 0 \]

gives

\[
\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin 3t\}
\]

\[
s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 4\mathcal{L}\{y\} = \frac{3}{s^2 + 9}
\]

\[
\mathcal{L}\{y\}(s^2 + 4) = \frac{3}{s^2 + 9}
\]

\[
\mathcal{L}\{y\} = \frac{3}{(s^2 + 9)(s^2 + 4)}
\]

We decompose the righthand side using partial fractions

\[
\frac{3}{(s^2 + 9)(s^2 + 4)} = \frac{As + B}{s^2 + 9} + \frac{Cs + D}{s^2 + 4}
\]

Multiplying through by \((s^2 + 9)(s^2 + 4)\) gives

\[ 3 = (As + B)(s^2 + 4) + (Cs + D)(s^2 + 9). \]

Expanding and collecting similar terms gives

\[ 3 = (A + C)s^3 + (B + D)s^2 + (4A + 9C)s + (4B + 9D). \]

Comparing coefficients on each side of the equation, we have two linear systems to solve. First, we see that \(A + C = 0\) and \(4A + 9C = 0\). The first of these equations gives \(A = -C\), which when plugged into the second equation, gives \(-4C + 9C = 0\), so \(C = 0\) and hence \(A = 0\).

Second, we also have \(B + D = 0\) and \(4B + 9D = 3\). The first of these equations gives \(B = -D\), which when plugged into the second equation, gives \(-4D + 9D = 3\), which means \(D = 3/5\). Since \(B = -D\), this means \(B = -3/5\). Therefore, we have

\[
\mathcal{L}\{y\} = \frac{3}{(s^2 + 9)(s^2 + 4)} = -\frac{3}{5} \cdot \frac{1}{s^2 + 9} + \frac{3}{5} \cdot \frac{1}{s^2 + 4}
\]

To invert the Laplace transform, note that \(\mathcal{L}\{\sin 3t\} = 3/(s^2 + 9)\) and \(\mathcal{L}\{\sin 2t\} = 2/(s^2 + 4)\). Therefore,

\[
\mathcal{L}\{y\} = -\frac{3}{5} \cdot \frac{1}{3} \cdot \frac{3}{s^2 + 9} + \frac{3}{5} \cdot \frac{1}{2} \cdot \frac{2}{s^2 + 4} = -\frac{1}{5} \cdot \frac{3}{s^2 + 9} + \frac{3}{10} \cdot \frac{2}{s^2 + 4}
\]

and the solution is

\[ y = -\frac{1}{5} \sin 3t + \frac{3}{10} \sin 2t \]