Math 2142 Homework 5 Part 1 Solutions

Problem 1. For the following homogeneous second order differential equations, give the general solution and the particular solution satisfying the given initial conditions.

$$y'' + 2y' - 3y = 0 \text{ with } y(0) = 6 \text{ and } y'(0) = -2$$
$$y'' + 4y' + 20y = 0 \text{ with } y(0) = 2 \text{ and } y'(0) = -8$$
$$y'' - 4y' + 4y = 0 \text{ with } y(0) = y'(0) = 1$$

Solution. For the first equation,

$$y'' + 2y' - 3y = 0 \text{ with } y(0) = 6 \text{ and } y'(0) = -2$$

the characteristic polynomial is $r^2 + 2r - 3 = (r+3)(r-1)$ with roots are $r = -3$ and $r = 1$. Therefore, the general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^x$$

To find the particular solution, we calculate $y'(x) = -3c_1 e^{-3x} + c_2 e^x$. Therefore, the initial conditions give

$$y(0) = 6 \Rightarrow 6 = c_1 + c_2$$
$$y'(0) = -2 \Rightarrow -2 = -3c_1 + c_2$$

Subtracting these equations gives $8 = 4c_1$ and so $c_1 = 2$. Therefore, $c_2 = 4$ and the particular solution is

$$y(x) = 2e^{-3x} + 4e^x$$

For the second equation,

$$y'' + 4y' + 20y = 0 \text{ with } y(0) = 2 \text{ and } y'(0) = -8$$

the characteristic polynomial is $r^2 + 4r + 20$. The roots are given by

$$r = \frac{-4 \pm \sqrt{4^2 - 4(1)(20)}}{2(1)} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

Therefore, the general solution is

$$y(x) = c_1 e^{-2x} \cos 4x + c_2 e^{-2x} \sin 4x$$

To find the particular solution, we plug in $y(0) = 2$ to get $2 = c_1$. To use the second condition we differentiate

$$\begin{aligned} y(x) &= 2e^{-2x} \cos 4x + c_2 e^{-2x} \sin 4x \\ y'(x) &= -4e^{-2x} \cos 4x - 8e^{-2x} \sin 4x - 2c_2 e^{-2x} \sin 4x + 4c_2 e^{-2x} \cos 4x \end{aligned}$$

and therefore

$$y'(0) = -4 + 4c_2$$

so $-8 = -4 + 4c_2$ and hence $c_2 = -1$. The particular solution is

$$y(x) = 2e^{-2x}\cos 4x - e^{-2x}\sin 4x$$

For the last equation

$$y'' - 4y' + 4y = 0 \text{ with } y(0) = y'(0) = 1$$

the characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$ and therefore there is a single (repeated) real root $r = 2$. The general solution is

$$y(x) = c_1e^{2x} + c_2xe^{2x}$$

The initial condition $y(0) = 1$ gives us that $1 = c_1$. Differentiating gives

$$\begin{aligned} y(x) &= e^{2x} + c_2xe^{2x} \\ y'(x) &= 2e^{2x} + c_2e^{2x} + 2c_2xe^{2x} \end{aligned}$$

The condition $y'(0) = 1$ gives us $1 = 2 + c_2$ and hence $c_2 = -1$. The particular solution is

$$y(x) = e^{2x} - xe^{2x}$$

Problem 2. Use the method of undetermined coefficients to find the general solutions for the following differential equations.

$$y'' - 5y' + 4y = e^{4x}$$
$$y'' + 3y' + 2y = x^2$$

Solution. To solve the first equation, we consider the homogeneous differential equation $y'' - 5y' + 4y = 0$. The characteristic polynomial is $r^2 - 5r + 4 = (r-4)(r-1)$ with roots $r = 4$ and $r = 1$. Therefore, the general solution to the homogeneous equation is $c_1e^x + c_2e^{4x}$.

To guess a solution to the non-homogeneous equation, we try $y = Axe^{4x}$ (since we know $y = Ae^{4x}$ is a solution to the homogeneous equation). Calculating the derivatives, we have

$$y' = Ae^{4x} + 4Axe^{4x}$$
$$y'' = 4Ae^{4x} + 4Ae^{4x} + 16Axe^{4x} = 8Ae^{4x} + 16Axe^{4x}$$

Plugging these values into the non-homogeneous equation, we have

$$y'' - 5y' + 4y = e^{4x}$$
$$(8Ae^{4x} + 16Axe^{4x}) - 5(Ae^{4x} + 4Axe^{4x}) + 4(Axe^{4x}) = e^{4x}$$
$$(8A - 5A)e^{4x} + (16A - 20A + 4A)xe^{4x} = e^{4x}$$
$$3Ae^{4x} = e^{4x}$$

and so $A = 1/3$. Therefore, one solution to the non-homogeneous equation is $y = 1/3\, xe^{4x}$ and the general solution to the non-homogeneous equation is

$$y = \frac{1}{3}xe^{4x} + c_1 e^x + c_2 e^{4x}$$

For the second equation, we consider the homogeneous differential equation $y'' + 3y' + 2y = 0$. The characteristic polynomial is $r^2 + 3r + 2 = (r+1)(r+2)$ with roots $r = -1$ and $r = -2$. Therefore, the general solution to the homogeneous equation is $c_1 e^{-x} + c_2 e^{-2x}$.

To guess a solution to the non-homogeneous equation, we try $y = Ax^2 + Bx + C$. Calculating the derivatives, we have

$$\begin{aligned} y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

Plugging these values into the non-homogeneous equation gives us

$$\begin{aligned} y'' + 3y' + 2y &= x^2 \\ 2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) &= x^2 \\ 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) &= x^2 \end{aligned}$$

Comparing the coefficients for the powers of x^2, we know $2A = 1$, so $A = 1/2$. Comparing the coefficients of x, we know $6A + 2B = 0$. Since $A = 1/2$, this equation tells us $3 + 2B = 0$ so $B = -3/2$. Finally, comparing the constant terms, we know $2A + 3B + 2C = 0$. Since $A = 1/2$ and $B = -3/2$, this equation tells us $1 - 9/2 + 2C = 0$ which means $C = 7/4$. Therefore, $y = x^2/2 - 3x/2 + 7/4$ is a single solution to the non-homogeneous equation and the general solution to the non-homogeneous equation is

$$y = x^2/2 - 3x/2 + 7/4 + c_1 e^{-x} + c_2 e^{-2x}$$

Problem 3. Use the method of undetermined coefficients to solve the initial value problems.

$$\begin{aligned} y'' + 4y' + 13y &= -3e^{-2x} \text{ with } y(0) = y'(0) = 0 \\ y'' + 6y' + 8y &= \cos x \text{ with } y(0) = y'(0) = 0 \end{aligned}$$

Solution. For the first equation, we consider the homogeneous equation $y'' + 4y' + 13y = 0$. The characteristic polynomial is $r^2 + 4r + 13$ which (using the quadratic formula) has roots $-2 \pm 3i$. Using $r = -2 + 3i$, we know that a complex solution to the homogeneous equation is given by $y = e^{-2x}\cos 3x + ie^{-2x}\sin 3x$ and therefore, the general (real) solution is $c_1 e^{-2x}\cos 3x + c_2 e^{-2x}\sin 3x$.

To guess a particular solution to the non-homogeneous equation, we try $y = Ae^{-2x}$. Calculating the derivatives, we have

$$\begin{aligned} y' &= -2Ae^{-2x} \\ y'' &= 4Ae^{-2x} \end{aligned}$$

Plugging these values into the non-homogeneous equation gives us

$$4Ae^{-2x} + 4(-2Ae^{-2x}) + 13Ae^{-2x} = -3e^{-2x}$$

which means $9A = -3$, so $A = -1/3$. Therefore, $y = -1/3e^{-2x}$ is a solution to the non-homogeneous equation and the general solution to the non-homogeneous equation is

$$y = -\frac{1}{3}e^{-2x} + c_1e^{-2x}\cos 3x + c_2e^{-2x}\sin 3x$$

To find the values of c_1 and c_2 corresponding to our initial conditions, we calculate (using the product and chain rules)

$$\begin{aligned} y' &= \frac{2}{3}e^{-2x} - 2c_1e^{-2x}\cos 3x - 3c_1e^{-2x}\sin 3x - 2c_2e^{-2x}\sin 3x + 3c_2e^{-2x}\cos 3x \\ y' &= \frac{2}{3}e^{-2x} + (-2c_1 + 3c_2)e^{-2x}\cos 3x + (-3c_1 - 2c_2)e^{-2x}\sin 3x \end{aligned}$$

To use $y(0) = 0$, we plug $x = 0$ into our general equation for y to get

$$-\frac{1}{3}e^0 + c_1e^0\cos 0 + c_2e^0\sin 0 = 0$$

which tells us that $c_1 = 1/3$. To use $y'(0) = 0$, we plug $x = 0$ into our general equation for y' to get

$$\frac{2}{3}e^0 + (-2c_1 + 3c_2)e^0\cos 0 + (-3c_1 - 2c_2)e^0\sin 0 = 0$$

which tells us that $-2c_1 + 3c_2 = -2/3$. Since $c_1 = 1/3$, we have $3c_2 = 0$ and so $c_2 = 0$. Therefore, the particular solution is

$$y = -\frac{1}{3}e^{-2x} + \frac{1}{3}e^{-2x}\cos 3x$$

For the second equation, we consider the homogeneous equation $y'' + 6y' + 8y = 0$. The characteristic equation is $r^2 + 6r + 8 = (r + 2)(r + 4)$ which has roots $r = -2$ and $r = -4$. Therefore, the general solution to the homogeneous equation is $y = c_1e^{-4x} + c_2e^{-2x}$.

To find a single solution to the non-homogeneous equation, we guess $y = A\sin x + B\cos x$. Calculating derivatives gives us

$$\begin{aligned} y' &= A\cos x - B\sin x \\ y'' &= -A\sin x - B\cos x \end{aligned}$$

Plugging these values into the non-homogeneous equation gives

$$\begin{gathered} y'' + 6y' + 8y = \cos x \\ -A\sin x - B\cos x + 6(A\cos x - B\sin x) + 8(A\sin x + B\cos x) = \cos x \\ (-A - 6B + 8A)\sin x + (-B + 6A + 8B)\cos x = \cos x \\ (7A - 6B)\sin x + (6A + 7B)\cos x = \cos x \end{gathered}$$

Setting the coefficients of $\cos x$ and $\sin x$ equal from the opposite sides of the equation, we have $7A - 6B = 0$ and $6A + 7B = 1$. Solving this system of equations yields $A = 6/85$ and $B = 7/85$. Therefore, the general solution to the non-homogeneous equation is

$$y = \frac{6}{85}\sin x + \frac{7}{85}\cos x + c_1 e^{-4x} + c_2 e^{-2x}$$

To find the specific solution, we use $y(0) = 0$ to get

$$\frac{6}{85}\sin 0 + \frac{7}{85}\cos 0 + c_1 e^0 + c_2 e^0 = 0$$

which means $c_1 + c_2 = -7/85$. To use $y'(0) = 0$, we first take the derivative

$$y' = \frac{6}{85}\cos x - \frac{7}{85}\sin x - 4c_1 e^{-4x} - 2c_2 e^{-2x}$$

and then use $y'(0) = 0$ by plugging in $x = 0$ to get

$$\frac{6}{85}\cos 0 - \frac{7}{85}\sin 0 - 4c_1 e^0 - 2c_2 e^0 = 0$$

to get $-4c_1 - 2c_2 = -6/85$. Solving $c_1 + c_2 = -7/85$ and $-4c_1 - 2c_2 = -6/85$ gives $c_1 = 2/17$ and $c_2 = -1/5$. Therefore, the specific solution to the non-homogeneous equation is

$$y = \frac{6}{85}\sin x + \frac{7}{85}\cos x + \frac{2}{17}e^{-4x} - \frac{1}{5}e^{-2x}$$