Math 2142 Homework 4 Part 2 Solutions

Problem 1. Prove that the set of functions $\{1, \sqrt{x}, x\}$ is independent.

Solution. We assume that
$$c_0 \cdot 1 + c_1\sqrt{x} + c_2 x = 0$$
and we need to show the that $c_0 = c_1 = c_2 = 0$. Plugging in $x = 0$ shows that $c_0 = 0$, so we are left with $c_1\sqrt{x} + c_2 x = 0$. Plugging in $x = 1$ and $x = 4$ gives the equations $c_1 + c_2 = 0$ and $2c_1 + 4c_2 = 0$. Since $c_1 + c_2 = 0$, we know $c_1 = -c_2$. Plugging this identity into $2c_1 + 4c_2 = 0$ gives $-2c_2 + 4c_2 = 0$, which means $c_2 = 0$. Since $c_1 = -c_2$, it also implies that $c_1 = 0$.

Problem 2. Prove that the set of functions $\{e^x, \sin x, \cos x\}$ is independent.

Solution. We assume that
$$c_0 e^x + c_1 \sin x + c_2 \cos x = 0$$
and we have to show that $c_0 = c_1 = c_2 = 0$. Plugging $x = 0$ and $x = \pi$ into the equation above gives
$$\begin{aligned} c_0 + c_2 &= 0 \\ c_0 e^\pi - c_2 &= 0 \end{aligned}$$
Adding these equations gives $c_0(1+\pi) = 0$ and so $c_0 = 0$. Since $c_0 + c_2 = 0$, we also have that $c_2 = 0$. We are left with $c_1 \sin x = 0$. Plugging in $x = \pi/2$ gives $c_1 = 0$, finishing the problem.

Alternately, once you have shown $c_0 = 0$, you are left with $c_1 \sin x + c_2 \cos x = 0$. In class, we proved that $\cos x$ and $\sin x$ are independent, so you can state this fact and immediately conclude that $c_1 = c_2 = 0$.

Problem 3. Prove that for each $n \geq 1$, the set of functions $\{1, x, x^2, \ldots, x^n\}$ is independent.

Solution. I will describe two ways to solve this problem. The first method is to do the proof by induction on $n \geq 1$. For the base case when $n = 1$, we have to show that $\{1, x\}$ is independent. We assume that $c_0 + c_1 x = 0$ and have to show that $c_0 = c_1 = 0$. Plugging $x = 0$ into $c_0 + c_1 x = 0$ gives us $c_0 = 0$ and leaves us with $c_1 x = 0$. Plugging in $x = 1$ gives us $c_1 = 0$.

For the induction case, we assume that $\{1, x, \ldots, x^n\}$ is independent and show that $\{1, x, \ldots, x^n, x^{n+1}\}$ is independent. We assume
$$c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + c_{n+1} x^{n+1} = 0$$
and have to show that $c_0 = c_1 = \cdots = c_n = c_{n+1} = 0$. First, if we plug in $x = 0$, we see that $c_0 = 0$. Second, if we take the derivative of the equation above, we get
$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + (n+1)c_{n+1} x^n = 0$$

Because $\{1, x, x^2, \ldots, x^n\}$ is independent, we know that each of the coefficients in this equation must be equal to 0. That is

$$c_1 = 0 \text{ and } 2c_2 = 0 \text{ and } 3c_3 = 0 \text{ and } \cdots \text{ and } nc_n = 0 \text{ and } (n+1)c_{n+1} = 0$$

It follows that $c_1 = 0, c_2 = 0, \ldots, c_n = 0, c_{n+1} = 0$.

The second method is a slightly less formal version of the same argument. To show $\{1, x, \ldots, x^n\}$ is independent, we assume that

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n = 0$$

We need to show that $c_0 = c_1 = \cdots = c_n = 0$. Plugging in $x = 0$ gives $c_0 = 0$. Taking the derivative gives us

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} = 0$$

and plugging in $x = 0$ tells us that $c_1 = 0$. Taking the derivative again gives us

$$2c_2 + 3 \cdot 2c_3 x + \cdots + n(n-1)c_n x^{n-2} = 0$$

and plugging in $x = 0$ tells us that $c_2 = 0$. Continuing in this manner of taking derivatives and plugging in $x = 0$, we eventually get down to

$$(n-1)!\, c_{n-1} + n(n-1)\cdots(3)(2)c_n x = 0$$

Plugging in $x = 0$ gives $c_{n-1} = 0$. Finally, taking one more derivative give

$$n!\, c_n = 0$$

from which it follows that $c_n = 0$.