Math 2142 Homework 4 Part 2: Due Friday February 16

In our study of differential equations, we used the notion of two functions being independent. We can extend this notion from two functions to a finite set of functions. A set $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ of functions is (*linearly*) *independent* if for any $c_0, \ldots, c_n \in \mathbb{R}$,

$$c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x) = 0 \text{ implies } c_0 = c_1 = \cdots = c_n = 0$$

The equality $c_0 f_0(x) + \cdots c_n f_n(x) = 0$ is between functions, so it holds for all values of x.

To show that a finite set of functions $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is independent, we typically proceed by assuming that $c_0 f_0(x) + \cdots c_n f_n(x) = 0$ and then showing that each $c_i = 0$.

In class, we used two methods to help show that a finite set of functions in independent. The first method is to plug in particular values of x to get linear equations involving the constants c_i. The second method is to take the derivative and use the fact that

$$c_0 f_0(x) + c_1 f_1(x) + \cdots + c_n f_n(x) = 0 \text{ implies } c_0 f_0'(x) + c_1 f_1'(x) + \cdots + c_n f_n'(x) = 0$$

Example. To show that the set of functions $\{1, x, x^2\}$ is independent, we begin with

$$\text{Assume: } c_0 \cdot 1 + c_1 x + c_2 x^2 = 0$$
$$\text{Show: } c_0 = c_1 = c_2 = 0$$

Plugging $x = 0$ into $c_0 + c_1 x + c_2 x^2$ gives $c_0 = 0$. So we are left with $c_1 x + c_2 x^2 = 0$ and we still need to show $c_1 = c_2 = 0$. There are a couple of ways we could proceed from here.

Method 1. If we plug $x = 1$ and $x = 2$ into $c_1 x + c_2 x^2 = 0$, we get $c_1 + c_2 = 0$ and $2c_1 + 4c_2 = 0$. The first equation tells us that $c_1 = -c_2$. Plugging $c_1 = -c_2$ into the second equation tells us that $-2c_2 + 4c_2 = 0$, so $c_2 = 0$. Then $c_1 + c_2 = 0$ and $c_2 = 0$ together tell us that $c_1 = 0$.

Method 2. Alternately, we could take the derivative of $c_1 x + c_2 x^2 = 0$ to get $c_1 + 2c_2 x = 0$. Plugging $x = 0$ into this equation gives us $c_1 = 0$. Since $c_1 x + c_2 x^2 = 0$ and $c_1 = 0$, we are left with just $c_2 x^2 = 0$. Plugging in $x = 1$ gives us that $c_2 = 0$.

Problem 1. Prove that the set of functions $\{1, \sqrt{x}, x\}$ is independent.

Problem 2. Prove that the set of functions $\{e^x, \sin x, \cos x\}$ is independent.

Problem 3. Prove that for each $n \geq 1$, the set of functions $\{1, x, x^2, \ldots, x^n\}$ is independent.