Math 2142 Homework 4 Part 1 Solutions

Problem 1. Solve the following separable initial value problems.

$$\frac{dy}{dx} = (1-2x)y^2 \qquad \text{with} \qquad y(0) = -1/6$$
$$\frac{dy}{dx} = xy^3(1+x^2)^{-1/2} \qquad \text{with} \qquad y(0) = 1$$

Solution. Consider the first initial value problem.

$$\frac{dy}{dx} = (1-2x)y^2 \qquad \text{with} \qquad y(0) = -1/6$$

Rewriting this separable equation, we get

$$\begin{aligned} \int \frac{1}{y^2}\,dy &= \int (1-2x)\,dx \\ \frac{-1}{y} &= x - x^2 + c \end{aligned}$$

Plugging in $y = -1/6$ and $x = 0$ from the initial condition gives $6 = c$. Solving for y gives

$$\begin{aligned} \frac{-1}{y} &= x - x^2 + 6 \\ \frac{1}{y} &= -x + x^2 - 6 \\ y &= \frac{1}{x^2 - x - 6} \end{aligned}$$

Now consider the second initial value problem.

$$\frac{dy}{dx} = xy^3(1+x^2)^{-1/2} \qquad \text{with} \qquad y(0) = 1$$

Rewriting this separable equation, we get

$$\begin{aligned} \int \frac{1}{y^3}\,dy &= \int \frac{x}{\sqrt{1+x^2}}\,dx \\ \frac{-1}{2y^2} &= \int \frac{x}{\sqrt{1+x^2}}\,dx \end{aligned}$$

To do the remaining integral, use the substitution $u = 1 + x^2$, so $du = 2x\,dx$. This gives

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + c = \sqrt{1+x^2} + c$$

Therefore, altogether we have

$$\frac{-1}{2y^2} = \sqrt{1+x^2} + c$$

Plugging in $y = 1$ and $x = 0$ from the initial condition gives $-1/2 = 1 + c$, so $c = -3/2$. To solve for y

$$\begin{aligned} \frac{-1}{2y^2} &= \sqrt{1+x^2} - \frac{3}{2} \\ \frac{1}{y^2} &= -2\sqrt{1+x^2} + 3 \\ y &= \frac{1}{\sqrt{3 - 2\sqrt{1+x^2}}} \end{aligned}$$

Note that we take the positive square root in the denominator because $y(0) = 1$.

Problem 2. Find the general solutions for the following separable differential equations. For the second one, you do not need to solve explicitly for y.

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \qquad \text{(you can assume that } x > 0\text{)}$$

$$\frac{dy}{dx} = \frac{2x}{ye^y - x^2ye^y} \qquad \text{(you can assume that } -1 < x < 1\text{)}$$

Solution. For the first differential equation

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}$$

we separate the variables to get

$$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx$$

The y integral is just $y^2/2$. To do the x integral, use the substitution $u = 1+x^3$, so $du = 3x^2\,dx$.

$$\int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|1+x^3| + c = \ln(\sqrt[3]{1+x^2}) + c$$

We can remove the absolute value signs because $x > 0$. Altogether, we have

$$\begin{aligned} y^2/2 &= \ln(\sqrt[3]{1+x^2}) + c \\ y &= \pm\sqrt{2\ln(\sqrt[3]{1+x^2}) + 2c} \end{aligned}$$

Note that we could replace the $2c$ in the second line by a new constant $\widehat{c}$.

For the second differential equation

$$\frac{dy}{dx} = \frac{2x}{ye^y - x^2ye^y}$$

we separate the variables to get

$$\int ye^y\,dy = \int \frac{2x}{1-x^2}\,dx$$

To calculate the y integral, we use integration by parts. Let $u = y$ and $dv = e^y\,dy$. Then $du = dy$ and $v = e^y$, giving us

$$\int ye^y\,dy = ye^y - \int e^y\,dy = ye^y - e^y$$

I am saving the integration constant for the end. To do the x integral, use the substitution $u = 1 - x^2$, so $du = -2x\,dx$.

$$\int \frac{2x}{1-x^2}\,dx = -\int \frac{1}{u}\,du = -\ln|1-x^2| + c = -\ln(1-x^2) + c$$

Note that we can remove the absolute value signs because $-1 < x < 1$. Altogether, we get

$$ye^y - e^y = -\ln(1-x^2) + c$$

which we will not attempt to solve for y!

Problem 3(a). Solve the following algebraic equation explicitly for y.

$$y^2 - 2y + 4 = x^4 + x^2 + 6$$

Solution. As indicated on the homework, we completing the square on the left side and solve for y.

$$\begin{aligned}
(y-1)^2 - 1 + 4 &= x^4 + x^2 + 6 \\
(y-1)^2 + 3 &= x^4 + x^2 + 6 \\
(y-1)^2 &= x^4 + x^2 + 3 \\
y &= \pm\sqrt{x^4 + x^2 + 3} + 1
\end{aligned}$$

3(b). Solve the following separable initial value problem and give the solution explicitly as a function $y(x)$.

$$\frac{dy}{dx} = \frac{3x^2 + e^x}{2y - 4} \qquad \text{with} \qquad y(0) = 1$$

Solution. Separating the variables, integrating and completing the square gives

$$\begin{aligned} \int 2y - 4\,dy &= \int 3x^2 + e^x\,dx \\ y^2 - 4y &= x^3 + e^x + c \\ (y-2)^2 - 4 &= x^3 + e^x + c \\ (y-2)^2 &= x^3 + e^x + \widehat{c} \\ y &= 2 \pm \sqrt{x^3 + e^x + \widehat{c}} \end{aligned}$$

where $\widehat{c} = c + 4$ is an altered constant of integration. To solve for the constant $\widehat{c}$, we plug in $y = 1$ and $x = 0$ to get

$$1 = 2 \pm \sqrt{0 + 1 + \widehat{c}}$$

It follows that $\widehat{c} = 0$ and we need to use the negative square root. So, the specific solution is

$$y = 2 - \sqrt{x^3 + e^x}$$

Problem 4. Find the general solution for the following linear differential equations. If an initial condition is given, find the specific solution as well.

$$\frac{dy}{dx} + 2xy = 4e^{-x^2} \quad \text{with} \quad y(0) = 3$$

$$\frac{dy}{dt} = \frac{y}{t} + 2t^2 \quad \text{for } t > 0$$

$$\frac{dy}{dx} + y = 3x$$

$$\frac{dy}{dx} + 3y = \sin 2x$$

Solution. For the first initial value problem

$$\frac{dy}{dx} + 2xy = 4e^{-x^2} \qquad \text{with} \qquad y(0) = 3$$

our integrating factor is

$$I(x) = e^{\int 2x\,dx} = e^{x^2}$$

The next integral we need to calculate is

$$\int I(x) \cdot Q(x)\,dx = \int e^{x^2} \cdot 4e^{-x^2}\,dx = \int 4\,dx = 4x + c$$

Since $1/I(x) = e^{-x^2}$, the general solution is

$$y(x) = e^{-x^2}(4x + c) = 4xe^{-x^2} + ce^{-x^2}$$

To find the constant c, we plug in $y = 3$ and $x = 0$ to get

$$3 = 4(0)e^0 + ce^0$$

Therefore, $c = 3$ and the specific solution is

$$y(x) = 4xe^{-x^2} + 3e^{-x^2}$$

Next, consider the second differential equation. We rewrite it in the form of a linear equation as

$$\frac{dy}{dt} - \frac{y}{t} = 2t^2$$

The integrating factor is

$$I(t) = e^{\int -1/t\,dt} = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$$

The next integral to calculate is

$$\int I(t) \cdot Q(t)\,dt = \int t^{-1} \cdot 2t^2\,dt = \int 2t\,dt = t^2 + c$$

Since $1/I(t) = t$, the general solution is

$$y(t) = t \cdot (t^2 + c) = t^3 + ct$$

Consider the third integral

$$\frac{dy}{dx} + y = 3x$$

The integrating factor is

$$I(x) = e^{\int 1\,dx} = e^x$$

The next integral we need to calculate is

$$\int I(x) \cdot Q(x)\,dx = \int e^x \cdot 3x\,dx = \int 3xe^x\,dx$$

We do this integral using integration by parts. Set $u = 3x$ and $dv = e^x\,dx$, so $du = 3\,dx$ and $v = e^x$. We get

$$\int 3xe^x\,dx = 3xe^x - \int 3e^x\,dx = 3xe^x - 3e^x + c = 3xe^x - 3e^x + c$$

(Notice that the c really should be inside the minus sign, but since it is a constant, it doesn't matter in the sense that it can be replaced by its negative.) Since $1/I(x) = e^{-x}$, the general solution is

$$y(x) = e^{-x}(3xe^x - 3e^x + c) = 3x - 3 + ce^{-x}$$

Finally, we consider the last differential equation

$$\frac{dy}{dx} + 3y = \sin 2x$$

The integration factor is

$$I(x) = e^{\int 3\,dx} = e^{3x}$$

The next integral to calculate is

$$\int I(x) \cdot Q(x)\,dx = \int e^{3x} \sin 2x\,dx$$

To do this integral, we will need to use integration by parts twice. First, let $u = \sin 2x$ and $dv = e^{3x}\,dx$, so $du = 2\cos 2x\,dx$ and $v = 1/3 \cdot e^{3x}$. We get

$$\int e^{3x} \sin 2x\,dx = 1/3 \cdot e^{3x} \sin 2x - 2/3 \cdot \int e^{3x} \cos 2x\,dx$$

To do the remaining integral, let $u = \cos 2x$ and $dv = e^{3x}\,dx$, so $du = -2\sin 2x\,dx$ and $v = 1/3 \cdot e^{3x}$. We get

$$\begin{aligned}
\int e^{3x} \sin 2x\,dx &= 1/3 \cdot e^{3x} \sin 2x - 2/3 \left(1/3 \cdot e^{3x} \cos 2x + 2/3 \cdot \int e^{3x} \sin 2x\,dx \right) \\
\int e^{3x} \sin 2x\,dx &= 1/3 \cdot e^{3x} \sin 2x - 2/9 \cdot e^{3x} \cos 2x - 4/9 \cdot \int e^{3x} \sin 2x\,dx \\
13/9 \cdot \int e^{3x} \sin 2x\,dx &= 1/3 \cdot e^{3x} \sin 2x - 2/9 \cdot e^{3x} \cos 2x \\
\int e^{3x} \sin 2x\,dx &= 3/13 \cdot e^{3x} \sin 2x - 2/13 \cdot e^{3x} \cos 2x + c
\end{aligned}$$

Note that I added the constant of integration in the last step. Since $1/I(x) = e^{-3x}$, the general solution is

$$y(x) = e^{-3x}(3/13 \cdot e^{3x} \sin 2x - 6/13 \cdot e^{3x} \cos 2x + c) = 3/13 \cdot \sin 2x - 2/13 \cdot \cos 2x + ce^{-3x}$$