Math 2142 Homework 3 Solutions

This homework comes in two parts. Please hand in the two parts on separate pieces of paper. The undergraduate grader will grade Part 1 and I will grade Part 2.

Homework 3 Part 1

Problem 1. From the textbook, Exercises 9.6, 1(a)-(h).

Solution. When using the formulas from class for division, we will often use the fact that $z \cdot \overline{z} = |z|^2$. That is, if $z = a + ib$, then $z \cdot \overline{z} = a^2 + b^2$. I will use this fact often in the following calculations. I will also use the facts that $i^2 = -1$, $i^3 = -i$ (since $i^3 = i^2 i$) and $i^4 = 1$ (since $i^4 = (i^2)^2 = (-1)^2$).

$$(1+i)^2 = 1^2 + 2i + i^2 = 1 + 2i - 1 = 2i$$
$$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = \frac{-i}{-i^2} = \frac{-i}{-(-1)} = -i$$
$$\frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{1^2+1^2} = \frac{1}{2} - i\,\frac{1}{2}$$
$$(2+3i)(3-4i) = 6 - 12i^2 - 8i + 9i = 6 + 12 + i = 18 + i$$
$$\frac{1+i}{1-2i} = \frac{1+i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{1-2+2i+i}{1^2+2^2} = \frac{-1}{5} + i\,\frac{3}{5}$$
$$i^5 + i^{16} = i^4 \cdot i + (i^4)^4 = 1 \cdot i + 1^4 = 1 + i$$
$$1 + i + i^2 + i^3 = 1 + i - 1 - i = 0$$
$$\frac{1}{2}(1+i)\left(1+i^{-8}\right) = \frac{1}{2}(1+i)\left(1+(i^4)^{-2}\right) = \frac{1}{2}(1+i)(1+1) = (1+i)$$

Problem 2. From the textbook, Exercises 9.6, 2(a)(c)(f).

Solution.

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$
$$\left|\frac{1+i}{1-i}\right| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{1^2+(-1)^2}} = 1$$
$$|2(1-i) + 3(2+i)| = |8+i| = \sqrt{8^2+1^2} = \sqrt{65}$$

Problem 3. From the textbook, Exercises 9.6, 3(a)(c)(e)(f).

Solution. The easiest way to do these problems is to think about where the points sit in the plane and give the polar coordinates. Remember that the principal argument θ has to satisfy $-\pi < \theta \leq \pi$. For 3(a), the number $2i$ sits at $(0, 2)$ and so have modulus 2 and $\theta = \pi/2$. For

3(c), the number -1 sits at $(-1,0)$ and so has modulus 1 and $\theta=\pi$. For 3(e), the number $-3+i\sqrt{3}$ sits at $(-3,\sqrt{3})$ and so has modulus $\sqrt{9+3}=\sqrt{12}=2\sqrt{3}$ and $\theta=5\pi/6$. (Draw the triangle in the plane.) For 3(f), the number $(1+i)/\sqrt{2}$ sits at $(1/\sqrt{2},1/\sqrt{2})$ and so has modulus 1 and $\theta=\pi/4$.

Problem 4. From the textbook, Exercises 9.10, 1(b)(e)(f)

Solution. First, I'll calculate these using the formula for the complex exponential.

$$\begin{aligned}
2e^{-\pi i/2} &= 2\,e^{0+i(-\pi/2)} = 2(e^0\cos(-\pi/2)+i\,e^0\sin(-\pi/2)) = 2(0+i(-1)) = -2i\\
i+e^{2\pi i} &= i+e^{0+i\,2\pi} = i+e^0\cos(2\pi)+i\,e^0\sin(2\pi) = i+1+0 = 1+i\\
e^{\pi i/4} &= e^{0+i\,\pi/4} = e^0\cos(\pi/4)+i\,e^0\sin(\pi/4) = 1/\sqrt{2}+i/\sqrt{2}
\end{aligned}$$

It is also worth thinking about these points geometrically because that is easier once you get used to working with complex numbers. Remember that each $z\in\mathbb{C}$ can be written as $z=|z|e^{i\theta}$ where θ is the argument (i.e. the polar angle).

For 1(b), we are given the point $2e^{-\pi i/2}$. From this form, we know that the modulus is 2 and the argument is $-\pi/2$. That, we want to point in the complex plane that corresponds to the point in $\mathbb{R}^2$ with polar coordinates $r=2$ and $\theta=-\pi/2$. This point is $(0,-2)$ in $\mathbb{R}^2$, which corresponds to the complex number $-2i$.

For 1(e), from the form of the complex number $e^{2\pi i}$, we see that it has modulus 1 and polar angle 2π. The point in $\mathbb{R}^2$ with polar coordinates $r=1$ and $\theta=2\pi$ is $(1,0)$. This point corresponds to the complex number 1, so $e^{2\pi i}=1$. Then $i+e^{2\pi i}=i+1$.

For 1(f), the form of $e^{\pi i/4}$ tells us that it corresponds to the point in $\mathbb{R}^2$ with polar coordinates $r=1$ and $\theta=\pi/4$. This point is $(1/\sqrt{2},1/\sqrt{2})$, and so the complex number is $1/\sqrt{2}+i/\sqrt{2}$.

Problem 5(a). Prove that for all $z_1,z_2\in\mathbb{C}$, we have $|z_1z_2|=|z_1|\,|z_2|$ and $\overline{z_1z_2}=\overline{z_1}\,\overline{z_2}$.

Solution. Let $z_1=a_1+ia_2$ and let $z_2=b_1+ib_2$. To show that $|z_1z_2|=|z_1|\,|z_2|$, it suffices to show that $|z_1z_2|^2=(|z_1|\,|z_2|)^2$ since all the quantities are positive. We calculate both sides to check that they are equal.

$$|z_1|^2|z_2|^2=(a_1^2+a_2^2)(b_1^2+b_2^2)=a_1^2b_1^2+a_1^2b_2^2+a_2^2b_1^2+a_2^2b_2^2$$

We next calculate z_1z_2.

$$z_1z_2=(a_1+ia_2)(b_1+ib_2)=(a_1b_1-a_2b_2)+i(a_1b_2+a_2b_1)$$

Therefore, $|z_1z_2|^2$ is given by

$$(a_1b_1-a_2b_2)^2+(a_1b_2+a_2b_1)^2=a_1^2b_1^2-2a_1a_2b_1b_2+a_2^2b_2^2+a_1^2b_2^2+2a_1a_2b_1b_2+a_2^2b_1^2$$

Canceling terms and comparing shows that $|z_1z_2|^2=(|z_1|\,|z_2|)^2$.

To see that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, we again calculate both sides. Using the formula for $z_1 z_2$ above, we have

$$\overline{z_1 z_2} = (a_1 b_1 - a_2 b_2) - i(a_1 b_2 + a_2 b_1)$$

Since $\overline{z_1} = a_1 - i a_2$ and $\overline{z_2} = b_1 - i b_2$, we get

$$\begin{aligned}\overline{z_1}\,\overline{z_2} &= (a_1 - i a_2)(b_1 - i b_2)\\ &= a_1 b_1 + i^2 a_2 b_2 - i a_1 b_2 - i a_2 b_1\\ &= (a_1 b_1 - a_2 b_2) - i(a_1 b_2 + a_2 b_1)\end{aligned}$$

Again, comparing these quantities shows that the equality holds.

5(b). Use 5(a) and induction to prove that for all $z \in \mathbb{C}$ and all natural numbers $n \geq 1$, we have $|z^n| = |z|^n$ and $\overline{z^n} = \overline{z}^n$.

Solution. The base case when $n = 1$ is trivial for both properties. For the induction case, assume that $|z^n| = |z|^n$ and $\overline{z^n} = \overline{z}^n$ for a fixed n and we show $|z^{n+1}| = |z|^{n+1}$ and $\overline{z^{n+1}} = \overline{z}^{n+1}$.

$$\begin{gathered}|z^{n+1}| = |z^n \cdot z| = |z^n| \cdot |z| = |z|^n \cdot |z| = |z|^{n+1}\\ \overline{z^{n+1}} = \overline{z^n \cdot z} = \overline{z^n} \cdot \overline{z} = \overline{z}^n \cdot \overline{z} = \overline{z}^{n+1}\end{gathered}$$

In each case, the second equality follows from 5(a) and the third equality uses the induction hypothesis.

Homework 3 Part 2

Problem 6(a). Let $r \in \mathbb{R}$ and $z \in \mathbb{C}$. Prove that $\overline{rz} = r\,\overline{z}$.

Solution. By Problem 5(a), $\overline{rz} = \overline{r}\,\overline{z}$, but since $r \in \mathbb{R}$, $\overline{r} = r$.

6(b). Let $p(z)$ be a polynomial with real coefficients. That is, $p(z)$ looks like

$$p(z) = r_0 + r_1 z + r_2 z^2 + \cdots + r_n z^n$$

with $r_0, r_1, \ldots, r_n \in \mathbb{R}$. Prove that $\overline{p(z)} = p(\overline{z})$.

Solution. Recall that in class we proved that for any $z_1, z_2 \in \mathbb{C}$, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. We use this property together with Problems 5 and 6(a) as follows:

$$\begin{aligned}\overline{p(z)} &= \overline{r_0 + r_1 z + r_2 z^2 + \cdots + r_n z^n}\\ &= \overline{r_0} + \overline{r_1 z} + \overline{r_2 z^2} + \cdots + \overline{r_n z^n} \quad \text{by the property from class}\\ &= r_0 + r_1 \overline{z} + r_2 \overline{z^2} + \cdots + r_n \overline{z^n} \quad \text{by Problem 6(a)}\\ &= r_0 + r_1 \overline{z} + r_2 \overline{z}^2 + \cdots + r_n \overline{z}^n \quad \text{by Problem 5}\\ &= p(\overline{z})\end{aligned}$$

6(c). Use 6(b) to explain why the non-real zeros of $p(z)$ must occur in conjugate pairs. That is, explain why if $p(z) = 0$ and z is not real, then $p(\overline{z}) = 0$ as well.

Solution. Suppose $z \in \mathbb{C}$ is not real and $p(z) = 0$. Taking the conjugate of both sides, we have $\overline{p(z)} = \overline{0}$. But, $\overline{0} = 0$ and by Problem 6(b), $\overline{p(z)} = p(\overline{z})$. Therefore, $p(\overline{z}) = 0$. (Notice that this calculation works just fine even if z is real, but in that case, $\overline{z} = z$ so we haven't found a new root of the polynomial p.)

Problem 7(a). Prove that if θ is real, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Solution. We use the fact that $e^{i\theta} = \cos\theta + i\sin\theta$ and that $e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$. Recall that sin is an odd function, so $\sin(-\theta) = -\sin\theta$, and cos is an even function, so $\cos(-\theta) = \cos\theta$. Therefore, $e^{-i\theta} = \cos\theta - i\sin\theta$. We can now calculate

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \frac{2\cos\theta}{2} = \cos\theta$$

7(b). Use 7(a) to prove that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.

Solution. First, we square both sides of the equation in Problem 7(a).

$$\cos^2\theta = \frac{(e^{i\theta} + e^{-i\theta})^2}{4} = \frac{(e^{i\theta})^2 + 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2}{4}$$

We can simplify by $(e^{i\theta})^2 = e^{i2\theta}$, $e^{i\theta}e^{-i\theta} = e^{i\theta - i\theta} = e^0 = 1$ and $(e^{-i\theta})^2 = e^{-i2\theta}$. Therefore, we have

$$\cos^2\theta = \frac{e^{i2\theta} + 2 + e^{-i2\theta}}{4} = \frac{e^{i2\theta} + e^{-i2\theta}}{4} + \frac{1}{2} = \frac{1}{2}\left(\frac{e^{i2\theta} + e^{-i2\theta}}{2} + 1\right)$$

Finally, notice that if we replace θ by 2θ in the formula from Problem 7(a), we get

$$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2}$$

Using this equality to substitute into the previous equation gives

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

as required.

Problem 8(a). Prove that if θ is real and n is a positive integer then

$$(\cos\theta + i\ \sin\theta)^n = \cos n\theta + i\ \sin n\theta$$

Solution. Using the identity $e^{i\theta} = \cos\theta + i\sin\theta$, we have

$$(\cos\theta + i\ \sin\theta)^n = (e^{i\theta})^n = e^{i\,n\theta}$$

But, $e^{i\,n\theta} = e^{0+i\,n\theta} = e^0\cos n\theta + ie^0\sin n\theta = \cos n\theta + i\sin n\theta$. Substituting into the equation above gives us the desired identity.

8(b). Use the case of $n = 3$ in 8(a) to prove that

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \qquad \text{and} \qquad \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

Solution. By the case of $n = 3$ in 8(a), we know that $(\cos\theta + i\ \sin\theta)^3 = \cos 3\theta + i\ \sin 3\theta$. Using the fact that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, we have

$$\begin{aligned}(\cos\theta + i\ \sin\theta)^3 &= \cos^3\theta + 3\cos^2\theta\cdot i\sin\theta + 3\cos\theta\cdot i^2\sin^2\theta + i^3\sin^3\theta \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Since $(\cos\theta + i\ \sin\theta)^3 = \cos 3\theta + i\ \sin 3\theta$, we have

$$\cos 3\theta + i\ \sin 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

Comparing the real and imaginary parts of this equations gives the desired trig identities.