## Math 2142 Homework 3: Due Friday February 9

This homework comes in two parts. Please hand in the two parts on separate pieces of paper. The undergraduate grader will grade Part 1 and I will grade Part 2.

### Homework 3 Part 1

**Problem 1.** From the textbook, Exercises 9.6, 1(a)-(h).

**Problem 2.** From the textbook, Exercises 9.6, 2(a)(c)(f).

**Problem 3.** From the textbook, Exercises 9.6, 3(a)(c)(e)(f).

**Problem 4.** From the textbook, Exercises 9.10, 1(b)(e)(f)

**Problem 5(a).** Prove that for all $z_1, z_2 \in \mathbb{C}$, we have $|z_1 z_2| = |z_1|\,|z_2|$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

**5(b).** Use 5(a) and induction to prove that for all $z \in \mathbb{C}$ and all natural numbers $n \geq 1$, we have $|z^n| = |z|^n$ and $\overline{z^n} = \overline{z}^n$.

### Homework 3 Part 2

If you use the properties of $\overline{z}$ that we proved in class and you proved in Problem 5, then the solutions to the first two parts of the next problem should be very short.

**Problem 6(a).** Let $r \in \mathbb{R}$ and $z \in \mathbb{C}$. Prove that $\overline{rz} = r\,\overline{z}$.

**6(b).** Let $p(z)$ be a polynomial with real coefficients. That is, $p(z)$ looks like

$$p(z) = r_0 + r_1\,z + r_2\,z^2 + \cdots + r_n\,z^n$$

with $r_0, r_1, \ldots, r_n \in \mathbb{R}$. Prove that $\overline{p(z)} = p(\overline{z})$.

**6(c).** Use 6(b) to explain why the non-real zeros of $p(z)$ must occur in conjugate pairs. That is, explain why if $p(z) = 0$ and $z$ is not real, then $p(\overline{z}) = 0$ as well.

One interesting feature of working with the exponential function in $\mathbb{C}$ is that it makes some trig identities fairly easy to prove. The next two problems will give you three examples of this phenomenon.

**Problem 7(a).** Prove that if $\theta$ is real, then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

*Hint.* Write $e^{i\theta}$ and $e^{-i\theta}$ in terms of their real and imaginary parts and then add them. How are $\cos\theta$ and $\cos -\theta$ related?

**7(b).** Use 7(a) to prove that $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$.

*Hint.* Square both sides of the equation in 7(a), then use 7(a) again when you are simplifying.

**Problem 8(a).** Prove that if $\theta$ is real and $n$ is a positive integer then

$$(\cos\theta + i\,\sin\theta)^n = \cos n\theta + i\,\sin n\theta$$

*Hint.* For what complex value $z$ is $e^z$ equal to $\cos\theta + i\,\sin\theta$? Once you find $z$, use the fact that $(e^z)^n = e^{nz}$.

**8(b).** Use the case of $n = 3$ in 8(a) to prove that

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta \qquad \text{and} \qquad \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

*Hint.* The case of $n = 3$ in 8(a) tells you that $(\cos\theta + i\,\sin\theta)^3 = \cos 3\theta + i\,\sin 3\theta$. Multiply out the lefthand side and then compare the real and imaginary parts of the two sides of the equation.

**Practice Problems**

Exercises 9.6: 2, 3, 4(a)(b)(c), 5(a)(b)(c)

Exercises 9.10: 1, 3, 4