Math 2142 Homework 2 Solutions

Problem 1. Prove the following formulas for Laplace transforms for $s > 0$.

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Solution. For the first Laplace transform, we need to calculate:

$$\int_0^\infty e^{-st} \cdot 1\, dt = \lim_{u\to\infty} \int_0^u e^{-st}\, dt = \lim_{u\to\infty} \frac{-e^{-st}}{s}\Big|_0^u = \lim_{u\to\infty} \frac{-e^{-su}}{s} + \frac{e^0}{s}$$

For each fixed value of $s > 0$, $e^{-su} \to 0$ as $u \to \infty$. Therefore, the first terms drops out and we are left with $e^0/s = 1/s$ as required.

For the second Laplace transform, we need to calculate:

$$\int_0^\infty e^{-st} \cdot t\, dt = \lim_{u\to\infty} \int_0^u e^{-st} t\, dt$$

To calculate $\int_0^u e^{-st} t\, dt$, we use integration by parts with $u = t$ and $dv = e^{-st}$. Therefore, $du = dt$ and $v = -e^{-st}/s$.

$$\begin{aligned}
\int_0^u e^{-st} t\, dt &= \frac{-te^{-st}}{s}\Big|_0^u + \frac{1}{s}\int_0^u e^{-st}\, dt \\
&= \frac{-ue^{-su}}{s} + 0 + \left(\frac{-e^{-st}}{s^2}\Big|_0^u\right) \\
&= \frac{-ue^{-su}}{s} + \left(\frac{-e^{-su}}{s^2} + \frac{1}{s^2}\right) \\
&= \frac{-u}{se^{su}} - \frac{1}{s^2 e^{su}} + \frac{1}{s^2}
\end{aligned}$$

By L'Hopital's Rule, for each fixed $s > 0$, $u/e^{su} \to 0$ as $u \to \infty$. Furthermore, for each fixed $s > 0$, $1/e^{su} \to 0$ as $u \to \infty$, Therefore, when we take the limit as $u \to \infty$ we are left with $1/s^2$ are required.

For the third Laplace transform, we need to calculate:

$$\int_0^\infty e^{-st} \cdot \sin at\, dt$$

Rather than dealing with the limit form of this integral, I will do integration by parts leaving ∞ as the upper limit. Let $u = e^{-st}$ and $dv = \sin at$, so $du = -se^{-st}$ and $v = (-1/a)\cos at$.

$$\int_0^\infty e^{-st} \cdot \sin at\, dt = \frac{-e^{-st}\cos at}{a}\Big|_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\, dt$$

Remember that the first term on the righthand side is an abbreviation for

$$\frac{-e^{-st}\cos at}{a}\bigg|_0^\infty = \lim_{u\to\infty}\frac{-e^{-st}\cos at}{a}\bigg|_0^u = \lim_{u\to\infty}\frac{-e^{-su}\cos au}{a} + \frac{e^0\cos 0}{a} = \lim_{u\to\infty}\frac{-\cos au}{ae^{su}} + \frac{1}{a}$$

Since $-1 \le \cos au \le 1$, we have $-1/e^{su} \le (\cos au)/e^{su} \le 1/e^{su}$. Since $s > 0$, $\lim_{u\to\infty} 1/e^{su} = 0$, so by the Squeeze theorem $\lim_{u\to\infty}(\cos au)/e^{su} = 0$. Therefore,

$$\frac{-e^{-st}\cos at}{a}\bigg|_0^\infty = \lim_{u\to\infty}\frac{-\cos au}{ae^{su}} + \frac{1}{a} = \frac{1}{a}$$

and we have

$$\int_0^\infty e^{-st}\cdot\sin at\,dt = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos at\,dt. \tag{1}$$

To calculate $\int_0^\infty e^{-st}\cos at\,dt$ we use parts again with $u = e^{-st}$ and $dv = \cos at$, so $du = -se^{-st}$ and $v = (1/a)\sin at$. Therefore,

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{e^{-st}\sin at}{a}\bigg|_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt.$$

Again, the first term on the righthand side is an abbreviation:

$$\frac{e^{-st}\sin at}{a}\bigg|_0^\infty = \lim_{u\to\infty}\frac{e^{-st}\sin at}{a}\bigg|_0^u = \lim_{u\to\infty}\frac{e^{-su}\sin au}{a} - \frac{e^0\sin 0}{a} = \lim_{u\to\infty}\frac{\sin au}{ae^{su}}$$

Because $-1 \le \sin au \le 1$, we have $1/e^{su} \le (\sin au)/e^{su} \le 1/e^{su}$. For $s > 0$, $\lim_{u\to\infty} 1/e^{su} = 0$ and so by the Squeeze theorem, $\lim_{u\to\infty}(\sin au)/e^{su} = 0$. Therefore,

$$\frac{e^{-st}\sin at}{a}\bigg|_0^\infty = \lim_{u\to\infty}\frac{\sin au}{ae^{su}} = 0$$

and we have

$$\int_0^\infty e^{-st}\cos at\,dt = \frac{s}{a}\int_0^\infty e^{-st}\sin at\,dt. \tag{2}$$

Putting Equations (1) and (2) together and doing some algebra, we have

$$\int_0^\infty e^{-st}\cdot\sin at\,dt = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin at\,dt$$
$$\left(1 + \frac{s^2}{a^2}\right)\int_0^\infty e^{-st}\cdot\sin at\,dt = \frac{1}{a}$$
$$\frac{a^2+s^2}{a^2}\int_0^\infty e^{-st}\cdot\sin at\,dt = \frac{1}{a}$$
$$\int_0^\infty e^{-st}\cdot\sin at\,dt = \frac{1}{a}\cdot\frac{a^2}{s^2+a^2} = \frac{a}{s^2+a^2}$$

which is the formula we are trying to prove.

For the last Laplace transform, we need to calculate:

$$\int_0^\infty e^{-st} \cdot \cos at \, dt$$

We could repeat similar calculations to those used in the last Laplace transform, but it will be shorter to use the formulas in Equations (1) and (2). Starting with Equation (2) and using Equation (1) to do a substitution, we have

$$\int_0^\infty e^{-st} \cos at \, dt = \frac{s}{a} \int_0^\infty e^{-st} \sin at \, dt = \frac{s}{a} \left(\frac{1}{a} - \frac{s}{a} \int_0^\infty e^{-st} \cos at \, dt \right) = \frac{s}{a^2} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \cos at \, dt.$$

We can produce the formula for $\mathcal{L}\{\cos at\}$ from here with some algebra.

$$\begin{aligned}
\int_0^\infty e^{-st} \cos at \, dt &= \frac{s}{a^2} - \frac{s^2}{a^2} \int_0^\infty e^{-st} \cos at \, dt \\
\left(1 + \frac{s^2}{a^2}\right) \int_0^\infty e^{-st} \cos at \, dt &= \frac{s}{a^2} \\
\frac{a^2 + s^2}{a^2} \int_0^\infty e^{-st} \cos at \, dt &= \frac{s}{a^2} \\
\int_0^\infty e^{-st} \cos at \, dt &= \frac{s}{a^2} \cdot \frac{a^2}{s^2 + a^2} = \frac{s}{s^2 + a^2}
\end{aligned}$$

which is the formula we are trying to prove.

Problem 2. Prove by induction on $n \in \mathbb{N}^+$ that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Solution. For the base case, we need to show that $\mathcal{L}\{t\} = 1/s^2$, which we did in Problem 1. For the induction case, assume that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

for a fixed value of n and we show that

$$\mathcal{L}\{t^{n+1}\} = \frac{(n+1)!}{s^{n+2}}$$

By the definition of the Laplace transform, we have

$$\mathcal{L}\{t^{n+1}\} = \int_0^\infty t^{n+1} e^{-st} \, dt$$

Apply integration by parts to this integral with $u = t^{n+1}$ and $dv = e^{-st}$, so $du = (n+1)t^n$ and $v = -e^{-st}/s$. Then we have

$$\mathcal{L}\{t^{n+1}\} = \frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty t^n e^{-st}\,dt \tag{3}$$

By the definition of the Laplace transform, we know that $\int_0^\infty t^n e^{-st}\,dt = \mathcal{L}\{t^n\}$. Therefore, we can substitute $\mathcal{L}\{t^n\}$ into Equation (3) and use the Inductive Hypothesis.

$$\begin{aligned}
\mathcal{L}\{t^{n+1}\} &= \frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty + \frac{n+1}{s}\int_0^\infty t^n e^{-st}\,dt \\
\mathcal{L}\{t^{n+1}\} &= \frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty + \frac{n+1}{s}\,\mathcal{L}\{t^n\} \\
\mathcal{L}\{t^{n+1}\} &= \frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty + \frac{n+1}{s}\cdot\frac{n!}{s^{n+1}} \\
\mathcal{L}\{t^{n+1}\} &= \frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty + \frac{(n+1)!}{s^{n+2}}
\end{aligned}$$

To finish the proof, we need to show that the first term on the righthand side of these equations is equal to 0. Writing this term out in full, we have

$$\frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty = \lim_{u\to\infty}\frac{-t^{n+1}e^{-st}}{s}\Big|_0^u = \lim_{u\to\infty}\frac{-u^{n+1}e^{-su}}{s} = \frac{-1}{s}\lim_{u\to\infty}\frac{u^{n+1}}{e^{su}}$$

Applying L'Hopital's rule $n+1$ many times shows that $\lim_{u\to\infty} u^{n+1}/e^{su} = 0$ and so

$$\frac{-t^{n+1}e^{-st}}{s}\Big|_0^\infty = 0$$

as required to finish the induction case.

Problem 3. Let $f(t)$ be a twice differentiable function such that f'' is continuous on $[0,\infty)$ and both f and f' have exponential order as $t\to\infty$. Fix constants K, M and a such that both $|f(t)| \le Ke^{at}$ and $|f'(t)| \le Ke^{at}$ for $t \ge M$. Prove that for $s > a$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Solution. As suggested in the hint, let $g(t) = f'(t)$. From class, we know that for $s > a$,

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Applying the formula for the Laplace transform of a derivative to $g(t)$, we have for $s > a$

$$\mathcal{L}\{g'(t)\} = s\,\mathcal{L}\{g(t)\} - g(0).$$

Substituting in the facts that $g(0) = f'(0)$ and $\mathcal{L}\{g(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$, we have

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{g'(t)\} = s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0) = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Problem 4. Calculate the following Taylor polynomials.

4(a). The 5th Taylor polynomial of $\ln x$ at $x = 1$.

Solution. Let $f(x) = \ln x$. Let c_n denote the coefficient of the n-th term in the Taylor polynomial for $f(x)$.

$$\begin{aligned}
f(x) = \ln x &\Rightarrow f(1) = 0 \Rightarrow c_0 = \frac{0}{0!} = 0 \\
f'(x) = \frac{1}{x} &\Rightarrow f'(1) = 1 \Rightarrow c_1 = \frac{1}{1!} = 1 \\
f''(x) = \frac{-1}{x^2} &\Rightarrow f''(1) = -1 \Rightarrow c_2 = \frac{-1}{2!} = -\frac{1}{2} \\
f'''(x) = \frac{2}{x^3} &\Rightarrow f'''(1) = 2 \Rightarrow c_3 = \frac{2}{3!} = \frac{1}{3} \\
f^{(4)}(x) = \frac{-2\cdot 3}{x^4} &\Rightarrow f^{(4)}(1) = -2\cdot 3 \Rightarrow c_4 = -\frac{2\cdot 3}{4!} = -\frac{1}{4} \\
f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{x^5} &\Rightarrow f^{(5)}(1) = 2\cdot 3\cdot 4 \Rightarrow c_5 = \frac{2\cdot 3\cdot 4}{5!} = \frac{1}{5}
\end{aligned}$$

Therefore, the 5th Taylor polynomial for $\ln x$ at $x = 1$ is

$$T_{5,1}\ln x = 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5$$

4(b). The 4th Taylor polynomial of $\sqrt{x}$ at $x = 1$.

Solution. Let $f(x) = \sqrt{x}$. Let c_n denote the coefficient of the n-th term in the Taylor polynomial for $f(x)$.

$$\begin{aligned}
f(x) = \sqrt{x} &\Rightarrow f(1) = 1 \Rightarrow c_0 = \frac{1}{0!} = 1 \\
f'(x) = \frac{1}{2}x^{-1/2} &\Rightarrow f'(1) = \frac{1}{2} \Rightarrow c_1 = \frac{1/2}{1!} = \frac{1}{2} \\
f''(x) = -\frac{1}{2^2}x^{-3/2} &\Rightarrow f''(1) = -\frac{1}{4} \Rightarrow c_2 = -\frac{1/4}{2!} = -\frac{1}{8} \\
f^{(3)}(x) = \frac{3}{2^3}x^{-5/2} &\Rightarrow f^{(3)}(1) = \frac{3}{8} \Rightarrow c_3 = \frac{3/8}{3!} = \frac{1}{16} \\
f^{(4)}(x) = -\frac{3\cdot 5}{2^4}x^{-7/2} &\Rightarrow f^{(4)}(1) = -\frac{15}{16} \Rightarrow c_4 = -\frac{15/16}{4!} = -\frac{5}{128}
\end{aligned}$$

Therefore, the 4th Taylor polynomial for $\sqrt{x}$ at $x = 1$ is

$$T_{4,1}\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

4(c). The 6th Taylor polynomial of $\cos x$ at $x = 0$.

Solution. Let $f(x) = \cos x$.

$$\begin{aligned} f(x) = \cos x \;\Rightarrow\; f(0) = 1 \;\Rightarrow\; c_0 = 1 \\ f'(x) = -\sin x \;\Rightarrow\; f'(0) = 0 \;\Rightarrow\; c_1 = 0 \\ f''(x) = -\cos x \;\Rightarrow\; f''(0) = -1 \;\Rightarrow\; c_2 = -\frac{1}{2!} \\ f'''(x) = \sin x \;\Rightarrow\; f'''(0) = 0 \;\Rightarrow\; c_3 = 0 \end{aligned}$$

At this point, the pattern of derivatives repeats, so we have $f^{(4)}(0) = \cos 0 = 1$, $f^{(5)}(0) = -\sin 0 = 0$ and $f^{(6)}(0) = -\cos 0 = -1$. Therefore, the 6th Taylor polynomial is

$$T_{6,0} \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Problem 5. Prove that the Taylor polynomial of degree $2n$ for $\cos(x)$ at $x = 0$ is

$$T_{2n}(\cos(x)) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k}$$

Solution. Let $f(x) = \cos x$. From the calculations in Problem 4(c), the pattern of values of $f^{(k)}(0)$ is that $f^{(k)}(0) = 0$ when k is odd and $f^{(k)}(0)$ alternates between 1 and -1 when k is even. Therefore, the Taylor polynomial is

$$T_{2n,0} \cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}\, x^{2k}$$

Problem 6. Let $f(x) = 1/(1-x) = (1-x)^{-1}$.

6(a). Prove by induction on k that $f^{(k)}(x) = k!\,(1-x)^{-(k+1)}$ and so $f^{(k)}(0) = k!$.

6(b). Prove that the n-th Taylor polynomial for $f(x)$ at $x = 0$ is

$$T_n f = 1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k$$

Solution. For 6(a), we proceed by induction. For the base case $k = 0$, we have $f^{(0)}(x) = f(x) = (1-x)^{-1}$ and $0!(1-x)^{-(0+1)} = (1-x)^{-1}$ which are equal. For the induction case, assume that $f^{(k)}(x) = k!\,(1-x)^{-(k+1)}$. We need to show that $f^{(k+1)}(x) = (k+1)!\,(1-x)^{-(k+2)}$.

$$f^{(k+1)}(x) = \frac{d}{dx}\, f^{(k)}(x) = \frac{d}{dx}\, k!\,(1-x)^{-(k+1)} = -k!\cdot(k+1)(1-x)^{-(k+1)-1}(-1) = (k+1)!(1-x)^{-(k+2)}.$$

For 6(b), since $f^{(k)}(0) = k!$, we have

$$T_{n,0}f = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{k!}{k!} x^k = \sum_{k=0}^{n} x^k$$

Problem 7(a). Use Problem 6 to find the n-th Taylor polynomial for $1/(1+x)$ at $x = 0$.

Solution. Since $1/(1+x) = 1/(1-(-x))$, we have by substitution that

$$\begin{aligned} T_{n,0}\frac{1}{1+x} &= \sum_{k=0}^{n}(-x)^k \\ &= 1 - x + x^2 - x^3 + x^4 - \cdots (-1)^n x^n \\ &= \sum_{k=0}^{n}(-1)^k x^k \end{aligned}$$

7(b). Use 7(a) to find the n-th Taylor polynomial for $1/(1+x)^2$ and the degree $2n$ Taylor polynomial for $1/(1+x^2)$, both also at $x = 0$.

Solution. Substituting x^2 for x, we get the $2n$-th Taylor polynomial

$$T_{2n,0}\frac{1}{1+x^2} = \sum_{k=0}^{n}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n}$$

Differentiating $1/(1+x)$ gives $-1/(1+x)^2$. Therefore,

$$\begin{aligned} T_{n,0}\frac{1}{(1+x)^2} &= -\frac{d}{dx} T_{n+1,0}\frac{1}{1+x} \\ &= -\frac{d}{dx}\sum_{k=0}^{n+1}(-1)^k x^k \\ &= -\sum_{k=0}^{n+1}\frac{d}{dx}(-1)^k x^k \\ &= -\sum_{k=1}^{n+1}(-1)^k k x^{k-1} \end{aligned}$$

Note that in the last sum, the bottom index shifted to $k = 1$ because the constant term in the previous line disappears when you take the derivative. To simplify this expression, note that $-(-1)^k$ can be rewritten as $(-1)^{k+1}$ or as $(-1)^{k-1}$. We shift indices as follows:

$$T_{n,0}\frac{1}{(1+x)^2} = \sum_{k=1}^{n+1}(-1)^{k-1} k x^{k-1} = \sum_{k=0}^{n}(-1)^k (k+1) x^k$$

7(c). Use 7(b) to find the degree $2n+1$ Taylor polynomial for $\arctan(x)$ at $x=0$.

Solution. Since $\arctan x = \int_0^x 1/(1+t^2)\,dt$, we can find the $(2n+1)$-st Taylor polynomial by integrating the formula for $T_{2n,0}\,1/(1+x^2)$.

$$T_{2n+1,0} \arctan x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Problem 8. Consider the operator defined by $D(f) = 2f' + f$.

8(a). Calculate $D(x)$, $D(x^2)$ and $D(3x^2 - 4x)$.

Solution. We calculate as follows.

$$\begin{aligned} D(x) &= 2(1) + x = 2 + x \\ D(x^2) &= 2(2x) + x^2 = 4x + x^2 \\ D(3x^2 - 4x) &= 2(6x - 4) + 3x^2 - 4x = 3x^2 + 8x - 8 \end{aligned}$$

8(b). Prove that D is linear. That is, show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any $\alpha, \beta \in \mathbb{R}$.

Solution.

$$\begin{aligned} D(\alpha f + \beta g) &= 2\frac{d}{dx}(\alpha f + \beta g) + (\alpha f + \beta g) \\ &= 2(\alpha f' + \beta g') + \alpha f + \beta g \\ &= (2\alpha f' + \alpha f) + (2\beta g' + \beta g) \\ &= \alpha(2f' + f) + \beta(2g' + g) \\ &= \alpha D(f) + \beta D(g) \end{aligned}$$

Problem 9. Consider the operator defined by $D(f) = f'' - 2f' + 3f$.

9(a). Calculate $D(x)$, $D(x^2)$ and $D(3x^2 - 4x)$.

Solution.

$$\begin{aligned} D(x) &= 0 - 2(1) + 3x = 3x - 2 \\ D(x^2) &= 2 - 2(2x) + 3x^2 = 3x^2 - 4x + 2 \\ D(3x^2 - 4x) &= 6 - 2(6x - 4) + 3(3x^2 - 4x) = 9x^2 - 24x + 14 \end{aligned}$$

9(b). Prove that D is linear.

Solution.

$$\begin{aligned} D(\alpha f + \beta g) &= \frac{d^2}{dx^2}(\alpha f + \beta g) - 2\frac{d}{dx}(\alpha f + \beta g) + 3(\alpha f + \beta g) \\ &= \alpha f'' + \beta g'' - 2(\alpha f' + \beta g') + 3\alpha f + 3\beta g \\ &= (\alpha f'' - 2\alpha f' + 3\alpha f) + (\beta g'' - 2\beta g' + 3\beta g) \\ &= \alpha(f'' - 2f' + 3f) + \beta(g'' - 2g' + 3g) \\ &= \alpha D(f) + \beta D(g) \end{aligned}$$

Problem 10. Consider the operator defined by $D(f) = e^x + f'$.

10(a). Calculate $D(x^2)$, $D(2x^2)$ and $2D(x^2)$.

Solution.

$$\begin{aligned} D(x^2) &= e^x + 2x \\ D(2x^2) &= e^x + 4x \\ 2D(x^2) &= 2(e^x + 2x) = 2e^x + 4x \end{aligned}$$

10(b). Use your answers to 10(a) to explain why D is not linear.

Solution. Since $D(2x^2) \neq 2\,D(x^2)$, we cannot pull multiplicative constants out of D, which means D is not linear.