Math 2142 Homework 2: Due Friday February 2

Problem 1. Prove the following formulas for Laplace transforms for $s > 0$.

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

Problem 2. Prove by induction on $n \in \mathbb{N}^+$ that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Problem 3. Let $f(t)$ be a twice differentiable function such that f'' is continuous on $[0, \infty)$ and both f and f' have exponential order as $t \to \infty$. Fix constants K, M and a such that both $|f(t)| \le Ke^{at}$ and $|f'(t)| \le Ke^{at}$ for $t \ge M$. Prove that for $s > a$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

Hint. This problem can be very short. Let $g(t) = f'(t)$. From our work in class, you know how $\mathcal{L}\{g'(t)\}$ is related to $\mathcal{L}\{g(t)\}$. Use this information to help you.

Problem 4. Calculate the following Taylor polynomials.

4(a). The 5th Taylor polynomial of $\ln x$ at $x = 1$.

4(b). The 4th Taylor polynomial of $\sqrt{x}$ at $x = 1$.

4(c). The 6th Taylor polynomial of $\cos x$ at $x = 0$.

Problem 5. Prove that the Taylor polynomial of degree $2n$ for $\cos(x)$ at $x = 0$ is

$$T_{2n}(\cos(x)) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$$

Problem 6. Let $f(x) = 1/(1-x) = (1-x)^{-1}$.

6(a). Prove by induction on k that $f^{(k)}(x) = k!\,(1-x)^{-(k+1)}$ and so $f^{(k)}(0) = k!$.

6(b). Prove that the n-th Taylor polynomial for $f(x)$ at $x = 0$ is

$$T_n f = 1 + x + x^2 + \cdots + x^n = \sum_{k=0}^{n} x^k$$

Problem 7(a). Use Problem 6 to find the n-th Taylor polynomial for $1/(1+x)$ at $x = 0$.

7(b). Use 7(a) to find the n-th Taylor polynomial for $1/(1+x)^2$ and the degree $2n$ Taylor polynomial for $1/(1+x^2)$, both also at $x = 0$.

7(c). Use 7(b) to find the degree $2n+1$ Taylor polynomial for $\arctan(x)$ at $x = 0$.

Problem 8. Consider the operator defined by $D(f) = 2f' + f$.

8(a). Calculate $D(x)$, $D(x^2)$ and $D(3x^2 - 4x)$.

8(b). Prove that D is linear. That is, show that $D(\alpha f + \beta g) = \alpha D(f) + \beta D(g)$ for any $\alpha, \beta \in \mathbb{R}$.

Hint. By definition, $D(\alpha f + \beta g) = 2(\alpha f + \beta g)' + (\alpha f + \beta g)$. Rewrite this expression until it turns into $\alpha D(f) + \beta D(g)$.

Problem 9. Consider the operator defined by $D(f) = f'' - 2f' + 3f$.

9(a). Calculate $D(x)$, $D(x^2)$ and $D(3x^2 - 4x)$.

9(b). Prove that D is linear.

Problem 10. Consider the operator defined by $D(f) = e^x + f'$.

10(a). Calculate $D(x^2)$, $D(2x^2)$ and $2D(x^2)$.

10(b). Use your answers to 10(a) to explain why D is not linear.

Practice Problems. Here is a list of additional problems of Taylor polynomials and their remainders that you can work on for more practice. You do not need to hand these in.

- Exercises 7.4: 5, 6, 7, 8, 10.
- Exercises 7.8: 1, 2, 3, 8.

Practice Problem 1. Let f be a function which is n times differentiable such that $f^{(n)}$ is continuous on $[0, \infty)$ and $f^{(k)}$ has exponential order as $t \to \infty$ for each $k < n$. Fix constants K, M and a such that $|f^{(k)}(t)| \leq Ke^{at}$ for all $k < n$. Prove that for $s > a$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

Hint. Prove this fact by induction following the model from Problem 3.