Math 2142 Homework 1 Solutions

Problem 1. Prove that for any $n \geq 2$, the log function $\ln(x)$ grows slower than $\sqrt[n]{x}$ by proving that

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$$

Solution. This limit has the form ∞/∞, so we apply L'Hopital's rule.

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[n]{x}} = \lim_{x\to\infty} \frac{x^{-1}}{\frac{1}{n}x^{(1/n)-1}} = \lim_{x\to\infty} \frac{nx^{-1}}{x^{1/n}x^{-1}} = \lim_{x\to\infty} \frac{n}{\sqrt[n]{x}} = 0.$$

The first equality is by L'Hopital's rule and the last equality follows because the denominator goes to infinity while the numerator is constant.

Problem 2. Find the following limits.

2(a).

$$\lim_{x\to\infty} \frac{3x^2 - x + 7}{e^x}$$

Solution. The limit has form ∞/∞. We will apply L'Hopital's rule twice.

$$\lim_{x\to\infty} \frac{3x^2 - x + 7}{e^x} = \lim_{x\to\infty} \frac{6x-1}{e^x} = \lim_{x\to\infty} \frac{6}{e^x} = 0.$$

The first and second equalities follow from L'Hopital's rule.

2(b).

$$\lim_{x\to\infty} \frac{(\ln(x))^2}{\sqrt[3]{x}}$$

Solution. The limit has form ∞/∞ so we apply L'Hopital's rule and do some algebra.

$$\lim_{x\to\infty} \frac{(\ln(x))^2}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{2\ln(x)\cdot\frac{1}{x}}{(1/3)x^{-2/3}} = \lim_{x\to\infty} \frac{6\ln x}{x^{1/3}}$$

This limit still has form ∞/∞ so we apply L'Hopital's rule again and do some algebra.

$$\lim_{x\to\infty} \frac{6\ln x}{x^{1/3}} = \lim_{x\to\infty} \frac{6/x}{(1/3)x^{-2/3}} = \lim_{x\to\infty} \frac{18}{x^{1/3}} = 0.$$

2(c).

$$\lim_{x\to\infty} \frac{\sin(x) - x}{x^3}$$

Solution. Since $-1 \leq \sin x \leq 1$ for all x, we have $\sin x - x \to -\infty$ as $x \to \infty$. Therefore, the limit has form ∞/∞ and we can apply L'Hopital's rule.

$$\lim_{x\to\infty} \frac{\sin(x) - x}{x^3} = \lim_{x\to\infty} \frac{\cos(x) - 1}{3x^2}$$

This limit does not have form ∞/∞ because the numerator satisfies $-2 \leq \cos(x) - 1 \leq 0$. Therefore, we cannot apply L'Hopital's rule again. However, we can use the Squeeze theorem. Since $-2 \leq \cos(x) - 1 \leq 0$, we have

$$\frac{-2}{3x^2} \leq \frac{\cos(x) - 1}{3x^2} \leq 0$$

Since $\lim_{x\to\infty} -2/3x^2 = 0$, we have $\lim_{x\to\infty}(\cos(x) - 1)/3x^2 = 0$.

2(d).

$$\lim_{x\to 0+} \sqrt{x}\ln(x)$$

Solution. This limit has form $0 \cdot \infty$ so we need to rewrite it before using L'Hopital's rule.

$$\lim_{x\to 0+} \sqrt{x}\ln(x) = \lim_{x\to 0+} \frac{\ln x}{x^{-1/2}}.$$

Now the limit has form ∞/∞ so we can apply L'Hopital's rule.

$$\lim_{x\to 0+} \frac{\ln x}{x^{-1/2}} = \lim_{x\to 0+} \frac{x^{-1}}{(-1/2)x^{-3/2}} = \lim_{x\to 0+} -2\sqrt{x} = 0.$$

2(e).

$$\lim_{x\to 0^+} \sin(x)\ln(x)$$

Solution. This limit has form $0 \cdot \infty$ so we need to rewrite it before using L'Hopital's rule.

$$\lim_{x\to 0^+} \sin(x)\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\csc x}$$

Now the limit has form ∞/∞ so we can apply L'Hopital's rule and simplify.

$$\lim_{x\to 0^+} \frac{\ln(x)}{\csc x} = \lim_{x\to 0^+} \frac{x^{-1}}{-\csc x \cot x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x}.$$

The limit now has form $0/0$ so we can apply L'Hopital's rule again.

$$\lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} = \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x - x\sin x} = \frac{-2(0)(1)}{1-(0)(0)} = 0.$$

2(f).

$$\lim_{x\to\infty} xe^{1/x} - x$$

Solution. This limit has form $\infty - \infty$ so we need to rewrite it.

$$\lim_{x\to\infty} xe^{1/x} - x = \lim_{x\to\infty} x(e^{1/x} - 1).$$

The limit now has form $\infty \cdot 0$ so we need to rewrite it as a fraction and apply L'Hopital's rule.

$$\lim_{x\to\infty} xe^{1/x} - x = \lim_{x\to\infty} \frac{e^{1/x}-1}{x^{-1}} = \lim_{x\to\infty} \frac{e^{1/x}\cdot(-1/x^2)}{-x^{-2}} = \lim_{x\to\infty} e^{1/x} = 1.$$

2(g).

$$\lim_{x\to 0^+} x^{\sqrt{x}}$$

Solution. This limit has the form 0^0 so we need to rewrite it using exponential and log functions.

$$\lim_{x\to 0^+} x^{\sqrt{x}} = \lim_{x\to 0^+} e^{\ln x^{\sqrt{x}}} = \lim_{x\to 0^+} e^{\sqrt{x}\ln x}$$

By Problem 2(d), we know that $\lim_{x\to 0^+} \sqrt{x}\ln x = 0$, so $\lim_{x\to 0^+} e^{\sqrt{x}\ln x} = 1$.

2(h).

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$$

Solution. This limit has the form 1^∞ so we need to rewrite it using exponential and log functions.

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{3}{x}\right)}$$

The limit in the exponent has form $\infty \cdot 0$, so we turn it into a fraction and use L'Hopital's rule.

$$\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{x^{-1}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}}\cdot\frac{-3}{x^2}}{-x^{-2}} = \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}} = 3.$$

Therefore, the final answer is e^3.

Problem 3. Calculate the following convergent integrals.

$$\int_1^\infty e^{-2x}\,dx$$

Solution. First, we calculate

$$\int_1^u e^{-2x}\,dx = (-1/2)e^{-2x}\Big|_1^u = (-1/2)e^{-2u} + (1/2)e^{-2}$$

Therefore,

$$\int_1^\infty e^{-2x}\,dx = \lim_{u\to\infty} (-1/2)e^{-2u} + (1/2)e^{-2} = 0 + (1/2)e^{-2} = \frac{1}{2e^2}.$$

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx$$

Solution. We calculate $\int_1^u \frac{\ln(x)}{x^2}\,dx$ by parts. Set $u = \ln x$ and $dv = x^{-2}\,dx$, which means $du = x^{-1}\,dx$ and $v = -x^{-1}$. Therefore,

$$\begin{aligned}
\int_1^u \frac{\ln(x)}{x^2}\,dx &= -\frac{\ln x}{x}\Big|_1^u - \int_1^u -\frac{1}{x}\cdot\frac{1}{x}\,dx \\
&= -\frac{\ln u}{u} + \int_1^u x^{-2}\,dx \\
&= -\frac{\ln u}{u} - \frac{1}{x}\Big|_1^u \\
&= -\frac{\ln u}{u} - \frac{1}{u} + 1
\end{aligned}$$

Since $\lim_{u\to\infty}(\ln u)/u = 0$ (by L'Hopital's rule), we have

$$\int_1^\infty \frac{\ln(x)}{x^2}\,dx = \lim_{u\to\infty} -\frac{\ln u}{u} - \frac{1}{u} + 1 = -0 - 0 + 1 = 1.$$

$$\int_0^3 \frac{1}{\sqrt{x}}\,dx$$

Solution. You did not have to do this problem, but I'll put in the solution since we will cover these types of integrals later. Since $1/\sqrt{x}$ has a vertical asymptote at $x = 0$, we have

$$\int_0^3 \frac{1}{\sqrt{x}}\,dx = \lim_{u\to 0^+}\int_u^3 x^{-1/2}\,dx = \lim_{u\to 0^+} 2x^{1/2}\Big|_u^3 = \lim_{u\to 0^+} 2\sqrt{3} - 2\sqrt{u} = 2\sqrt{3}.$$

Problem 4. Use the Comparison Theorem to determine if the following integrals converge or diverge. You do *not* need to calculate the exact value of the integrals.

$$\int_e^\infty \frac{\ln(x)}{x}\,dx$$

Solution. For $x \geq e$, we know that $\ln x \geq 1$. Therefore, $(\ln x)/x \geq 1/x$. Since $\int_e^\infty 1/x\,dx$ diverges, this integral diverges as well.

$$\int_1^\infty \frac{1}{x + e^{2x}}\,dx$$

Solution. Since $0 \leq 1/(x + e^{2x}) \leq 1/e^{2x}$ and $\int_1^\infty 1/e^{2x}\,dx$ converges (from Problem 2), this integral converges as well.

$$\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$$

Solution. Since $0 \le x/\sqrt{x^6} \le x/\sqrt{1+x^6}$ and $x/\sqrt{x^6} = 1/x^2$ and $\int_1^\infty 1/x^2\,dx$ converges, this integral converges as well.

$$\int_2^\infty \frac{x+2}{\sqrt{x^3-1}}\,dx$$

Solution. Notice that we have the following inequalities for $x \ge 2$.

$$0 \le \frac{x}{\sqrt{x^3}} \le \frac{x}{\sqrt{x^3-1}} \le \frac{x+2}{\sqrt{x^3-1}}$$

Since $x/\sqrt{x^3} = 1/x^{1/2}$ and $\int_2^\infty 1/x^{1/2}\,dx$ diverges (by the p-test since $p = 1/2 \le 1$), this integral diverges as well.

$$\int_1^\infty \frac{\sin^2 x}{1+x^2}\,dx$$

Solution. We have the following inequalities for $x \ge 1$.

$$0 \le \frac{\sin^2 x}{1+x^2} \le \frac{1}{1+x^2} \le \frac{1}{x^2}$$

Since $\int_1^\infty 1/x^2\,dx$ converges, this integral converges as well.