Math 2142 Homework 10: Due Friday April 20

Problem 1. In Exercise 12.15, do Exercises 2, 3, 4, 9 and 10.

Problem 2. For each of the following sets of vectors, decide if the set is linearly independent or not. If the set is linearly dependent, write $\overline{0}$ as a linear combination of the vectors in which at least one coefficient is non-zero.

$$S_0 = \{\langle 2,3\rangle, \langle -1,2\rangle\}$$
$$S_1 = \{\langle 0,2,-1\rangle, \langle 2,2,0\rangle, \langle 6,16,-5\rangle\}$$
$$S_2 = \{\langle 1,-2,1\rangle, \langle -4,8,7\rangle, \langle 2,-9,0\rangle\}$$

Problem 3(a). Show that $S = \{\langle 1,0,1\rangle, \langle -1,1,1\rangle, \langle 1,2,-1\rangle\}$ is a basis for $\mathbb{R}^3$.

3(b). Write $\langle 3,0,-4\rangle$ as a linear combination of the vectors in S.

For the next problems, recall that a set $S = \{\overline{u}_1, \ldots, \overline{u}_k\}$ of vectors in $\mathbb{R}^n$ is an *orthogonal set* if $\overline{u}_i \cdot \overline{u}_j = 0$ whenever $i \neq j$. In class, we proved that if S is an orthogonal set of non-zero vectors, then S is linearly independent. You can use this fact in the problems below.

Problem 4(a). Let $\langle a,b\rangle$ be a non-zero vector in $\mathbb{R}^2$. Prove that $S = \{\langle a,b\rangle, \langle -b,a\rangle\}$ is an orthogonal set.

4(b). Explain why S is a basis for $\mathbb{R}^2$.

Problem 5(a). Prove that $S = \{\langle 1,0,1\rangle, \langle -1,1,1\rangle, \langle 1,2,-1\rangle\}$ is an orthogonal set.

5(b). Explain why S is a basis for $\mathbb{R}^3$.

5(c). Explain why you know $\langle 1,2,-1\rangle$ is not in $\text{Span}(\{\langle 1,0,1\rangle, \langle -1,1,1\rangle\})$. (Hint. You do not need to do any calculations to answer this question.)

Problem 6. Let $S = \{\overline{u}_1, \ldots, \overline{u}_n\}$ be an orthogonal basis for $\mathbb{R}^n$. That is, S is an orthogonal set of vectors which is a basis for $\mathbb{R}^n$. Since S is a basis, you know that for any vector $\overline{V} \in \mathbb{R}^n$, there are scalars $c_1, \ldots, c_n$ such that

$$\overline{v} = c_1\overline{u}_1 + c_2\overline{u}_2 + \cdots + c_n\overline{u}_n$$

Prove that for each index i, we have

$$c_i = \frac{\overline{v} \cdot \overline{u}_i}{\|\overline{u}_i\|^2}$$

Hint. Use a method similar to our proof that an orthogonal set of non-zero vectors is linearly independent. For $\overline{v} = c_1\overline{u}_1 + c_2\overline{u}_2 + \cdots + c_n\overline{u}_n$, take the dot product of each side with $\overline{u}_i$. Simplify the righthand side and solve for c_i.

Problem 7. Let $S = \{\langle 1,0,1\rangle, \langle -1,1,1\rangle, \langle 1,2,-1\rangle\}$ be the orthogonal basis from Problem 5. Use the method in Problem 6 to write $\langle 2,-1,6\rangle$ and $\langle 0,12,2\rangle$ as linear combinations of the vectors in S.

Problem 8. For the point $P = \langle 1,0\rangle$ and vector $\overline{v} = \langle -2,3\rangle$, write the equation for the line through P with direction $\overline{v}$ in vector form, in parametric form and in normal form. For the normal form, give the answer in the form $ax + by = c$.

Problem 9. For the point $P = \langle 1,3,-1\rangle$ and the direction vectors $\overline{u} = \langle 2,0,1\rangle$ and $\overline{v} = \langle 3,-2,1\rangle$, write the equation for the plane through P with direction vectors $\overline{u}$ and $\overline{v}$ in vector form, parametric form and normal form. For the normal form, give the answer in the form $ax + by + cz = d$.

Problem 10. Find the area of the triangle with vertices $\langle 0,2,2\rangle$, $\langle 2,0,-1\rangle$ and $\langle 3,4,0\rangle$.

Hint. The area of this triangle is half the area of a parallelogram that you can use the cross product to find the area of.

Problem 11. Let $S = \{\overline{u}, \overline{v}\} \subseteq \mathbb{R}^3$ be a linearly independent set. Prove that $\{\overline{u}, \overline{v}, \overline{u} \times \overline{v}\}$ is a basis for $\mathbb{R}^3$.

Practice Problems

If you want some extra practice, you can try the following problems from the textbook.

- 12.4: 3, 8.
- 12.8: 2, 4, 7, 8, 10.
- 12.11: 2, 3.
- 12.15: 1, 4, 13, 17.
- 13.5: 5, 8.
- 13.8: 3, 4, 5, 6.
- 13.11 1, 2, 3, 5.