Math 2142 Final Exam Review Problems

Problem 1. Determine whether the following sets of vectors form a basis for $\mathbb{R}^2$. If they do not form a basis, explain why not. In particular, if they are not independent, give a nontrivial linear combination for $\overline{0}$ and if they do not span $\mathbb{R}^2$, describe the span.

$$\{\langle 1,2\rangle, \langle -2,-4\rangle\}$$
$$\{\langle 3,-2\rangle, \langle 1,4\rangle\}$$
$$\{\langle -1,1\rangle, \langle 0,0\rangle\}$$
$$\{\langle 2,3\rangle, \langle -1,4\rangle\}$$
$$\{\langle 1,5\rangle, \langle -1,2\rangle, \langle 3,6\rangle\}$$

Problem 2. Determine whether the following sets of vectors form a basis for $\mathbb{R}^3$. If they do not form a basis, find a nontrivial linear combination for $\overline{0}$ and give a vector which is not in the span.

$$\{\langle 3,0,4\rangle, \langle 2,3,2\rangle, \langle 0,5,-1\rangle\}$$
$$\{\langle 4,-2,-2\rangle, \langle 3,-2,-3\rangle, \langle -5,4,7\rangle\}$$
$$\{\langle 5,0,0\rangle, \langle 7,2,-6\rangle, \langle 9,4,-8\rangle\}$$

Problem 3. Find the area of the triangle with vertices $P=(1,2,3)$, $Q=(2,4,5)$ and $R=(0,3,4)$.

Hint. Let A be the vector from P to Q and let B be the vector from P to R. You know that $\|A \times B\|$ gives you the area of the parallelogram formed by these vectors. How is the area of the triangle related to the area of the parallelogram?

Problem 4. For each of the following points P and vectors $\overline{v}$, write the equation for the line through P with direction $\overline{v}$ in both vector form and parametric form.

$$P = \langle 2,1\rangle \text{ and } \overline{v} = \langle -1,5\rangle$$
$$P = \langle 1,0,-3\rangle \text{ and } \overline{v} = \langle 2,2,-1\rangle$$
$$P = \langle 0,0,0,0\rangle \text{ and } \overline{v} = \langle 2,3,-5,1\rangle$$

Problem 5(a). Let $A = \langle a,b\rangle$ be a nonzero vector in $\mathbb{R}^2$. Prove that $B = \langle -b,a\rangle$ is orthogonal to A.

5(b). Prove that $\{A,B\}$ is a basis for $\mathbb{R}^2$.

5(c). Explain why B cannot be in the span of $\{A\}$.

Problem 6. For each of the following points P and vectors $\overline{v}$, write the equation for the line in $\mathbb{R}^2$ through P with direction vector $\overline{v}$ in vector form, parametric form and normal form.

For the normal form, give the final answer in the form $ax + by = c$.

$$P = \langle 2, 1 \rangle \text{ and } \overline{v} = \langle -1, 5 \rangle$$
$$P = \langle -1, 4 \rangle \text{ and } \overline{v} = \langle 0, 2 \rangle$$
$$P = \langle 0, 0 \rangle \text{ and } \overline{v} = \langle 6, 1 \rangle$$

Hint. Use Problem 5(a) to help you find the normal vector for the line to write the equations in normal form.

Problem 7. For each of the following points P and vectors A and B, write the equation for the plane in $\mathbb{R}^3$ through P with direction vectors A and B in vector form, parametric form and normal form. For the normal form, give the final answer in the form $ax + by + cz = d$.

$$P = \langle 2, 0, 1 \rangle \text{ with } A = \langle 1, 1, 3 \rangle \text{ and } B = \langle -4, 2, 1 \rangle$$
$$P = \langle -1, 4, 2 \rangle \text{ with } A = \langle 0, 2, -3 \rangle \text{ and } B = \langle 1, 1, -5 \rangle$$
$$P = \langle 0, 0, 0 \rangle \text{ with } A = \langle 1, 0, 1 \rangle \text{ and } B = \langle 2, 1, -1 \rangle$$

Problem 8. Let $S = \{\overline{v}_1, \ldots, \overline{v}_k\}$ be an independent set of vectors in $\mathbb{R}^n$ and assume that $\overline{u}$ is not in the span of S. Prove that $S \cup \{\overline{u}\}$ is independent.

Hint. Suppose that $c_1\overline{v}_1 + \cdots + c_k\overline{v}_k + d\overline{u} = \overline{0}$. First, explain why $d = 0$. Second, once you know $d = 0$, explain why $c_1 = \cdots = c_k = 0$.

Problem 9. Let $S = \{A, B\}$ be a linearly independent set in $\mathbb{R}^3$. Prove that $A \times B$ is not in the span of S.

Problem 10. Let $S = \{\langle 1, 0, 1 \rangle, \langle 2, 1, 5 \rangle\}$. Prove that S is linearly independent and find a vector $\overline{v}$ such that $S \cup \{\overline{v}\}$ is a basis for $\mathbb{R}^3$.

Hint. To find $\overline{v}$, you need to find a vector which is not in the span of S. Use Problem 9 to help you.

Problem 11. Prove the following properties of the cross product.

$$A \times B = -(B \times A) \qquad r\,(A \times B) = (rA) \times B \qquad A \times (B + C) = (A \times B) + (A \times C)$$

Problem 12. Let A and B be linearly independent vectors in $\mathbb{R}^3$. Prove that $\{A, B, A \times B\}$ is a basis for $\mathbb{R}^3$.

Problem 13(a). Let $A = \langle a, b \rangle$ and $B = \langle c, d \rangle$ be vectors in $\mathbb{R}^2$. Prove that A and B are linearly independent if and only if $ad - bc \neq 0$.

13(b). Let A and B are vectors in $\mathbb{R}^2$. Prove that $\{A, B\}$ is a basis for $\mathbb{R}^2$ if and only if the matrix with rows A and B has a nonzero determinant.

Problem 14. Let A_1 and A_2 be nonzero orthogonal vectors. Prove that $\{A_1, A_2\}$ is linearly independent.

Problem 15(a). Let A and B be vectors in $\mathbb{R}^n$. Prove that

$$\|A - B\|^2 = \|A\|^2 + \|B\|^2 - 2\|A\|\|B\| \cos\theta$$

where θ is the angle between the vectors A and B.

15(b). Assume that the vectors A and B are independent so they form two sides of a triangle. Interpret the result in 15(a) to give you the Law of Cosines.

Problem 16. Let $\{A, B\}$ be an independent set in $\mathbb{R}^n$ with $\|A\| = \|B\|$. Because their lengths are equal, the vectors A and B form the sides of a rhombus. Prove that the diagonals of the rhombus are orthogonal. (*Hint.* Start by writing the diagonals in vector form.)

Problem 17. Let $\{A, B\}$ be an independent set of vectors and let $r, s \in \mathbb{R}$ be fixed nonzero scalars. Prove that $\text{Span}\{A, B\} = \text{Span}\{A, rA + sB\}$.

Hint. First, show $\text{Span}\{A, rA + sB\} \subseteq \text{Span}\{A, B\}$. To do this, let $C = c_0A + c_1(rA + sB)$ be a linear combination of the vectors in $\{A, rA + sB\}$. You need to rewrite C as a linear combination of A and B. That is, show that $C = d_0A + d_1B$ for some $d_0, d_1 \in \mathbb{R}$.

Second, show $\text{Span}\{A, B\} \subseteq \text{Span}\{A, rA + sB\}$. To do this containment, I would start by showing that A and B are in $\text{Span}\{A, rA + sB\}$. That is, find d_0 and d_1 such that $A = d_0A + d_1(rA + sB)$. Then find d_0 and d_1 such that $B = d_0A + d_1(rA + sB)$. Finally, consider a general linear combination $c_0A + c_1B$ and show there are d_0 and d_1 such that $c_0A + c_1B = d_0A + d_1(rA + sB)$.

Problem 18(a). Consider the independent set $\{\langle 1, 2, 3\rangle, \langle 1, -1, 1\rangle\}$. In this problem, you will find a vector $\overline{u}$ so that $\langle 1, 2, 3\rangle$ is orthogonal to $\overline{u}$ but $\text{Span}\{\langle 1, 2, 3\rangle, \langle 1, -1, 1\rangle\} = \text{Span}\{\langle 1, 2, 3\rangle, \overline{u}\}$. Use the following steps.

- Find the projection P of $\langle 1, -1, 1\rangle$ onto $\langle 1, 2, 3\rangle$ and calculate $\overline{u} = \langle 1, -1, 1\rangle - P$.
- Prove that $\langle 1, 2, 3\rangle$ and $\overline{u}$ are orthogonal.
- Prove that $\text{Span}\{\langle 1, 2, 3\rangle, \langle 1, -1, 1\rangle\} = \text{Span}\{\langle 1, 2, 3\rangle, \overline{u}\}$.

Hint. When showing that $\text{Span}\{\langle 1, 2, 3\rangle, \langle 1, -1, 1\rangle\} = \text{Span}\{\langle 1, 2, 3\rangle, \overline{u}\}$ use Problem 17 to help you. That is, how is $\overline{u}$ related to $\langle 1, 2, 3\rangle$ and $\langle 1, -1, 1\rangle$? You should be able to do this part with almost no work once you see how Problem 17 helps you.

18(b). Prove that this process works in general. Let $\{\overline{v}_0, \overline{v}_1\}$ be an independent set of vectors.

- Let P be the projection of $\overline{v}_1$ onto $\overline{v}_0$ and let $\overline{u} = \overline{v}_1 - P$. (Leave P in the form $r\overline{v}_0$ for an appropriate scalar r. Then $\overline{u} = \overline{v}_1 - r\overline{v}_0$.)
- Prove that $\overline{v}_0$ is orthogonal to $\overline{u}$. (*Hint.* Keep $\overline{u}$ written in form $\overline{v}_1 - r\overline{v}_0$. Good things will happen when you take the dot product $\overline{v}_0 \cdot \overline{u}$.)
- Prove that $\text{Span}\{\overline{v}_0, \overline{v}_1\} = \text{Span}\{\overline{v}_0, \overline{u}\}$.

Exercises from textbook for practice.

- 12.4: 3, 8
- 12.8: 2, 4, 7, 8, 10
- 12.11: 2, 3
- 12.15: 1, 4, 13, 17
- 13.5: 5, 8
- 13.8: 3, 4, 5, 6
- 13.11: 1, 2, 3, 5
- 13.14: 1, 2, 3, 4
- 13.17: 1, 5(a)(b), 6