Math 2142 Exam 1 Review Problems

Problem 1. Calculate the 3rd Taylor polynomial for $\arcsin x$ at $x = 0$.

Problem 2. In this problem, you will calculate the n-th Taylor polynomial for $x^{-1/2}$ at $x = 1$. To make the notation easier, let $f(x) = x^{-1/2}$.

2(a). Prove by induction that for all $n \geq 1$,

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \, x^{-(2n+1)/2}$$

2(b). Give the formula for the n-th Taylor polynomial to $x^{-1/2}$ at $x = 1$.

Problem 3. In this problem, you will develop the Taylor polynomials for $\ln(1-x)$. Let $f(x) = \ln(1-x)$.

3(a). Prove by induction that for all $n \geq 1$,

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}$$

and hence for $n \geq 1$, $f^{(n)}(0) = -(n-1)!$.

3(b). Give the n-th Taylor polynomial for $f(x)$ at $x = 0$.

Problem 4. Prove that if c is a constant and f is a (sufficiently differentiable) function, then

$$T_n(cf(x)) = c\,T_n(f(x))$$

where we take the Taylor polynomial at $x = a$.

Problem 5. Evaluate the following (convergent) improper integrals.

$$\int_4^\infty \frac{1}{(3x+1)^2}\,dx$$

$$\int_2^\infty e^{-x/2}\,dx$$

$$\int_{-\infty}^\infty e^{-|x|}\,dx$$

Problem 6. Use the definition of the improper integral to explain why $\int_0^\infty \sin x\,dx$ diverges.

Problem 7. Determine whether the following integrals converge or diverge. You do not need to calculate the value of the convergent integrals.

$$\int_1^\infty \frac{x^2}{9+x^6}\,dx$$
$$\int_1^\infty \frac{2+e^{-x}}{x}\,dx$$
$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$$
$$\int_2^\infty \frac{\ln x}{x^2}\,dx$$
$$\int_1^\infty \frac{x}{1+x^2}\,dx$$

For the last two, the initial comparisons you might think of to integrals of the form $\int_1^\infty 1/x^p\,dx$ go the wrong way. Try altering the $1/x^p$ slightly to make the comparison work.

Problem 8. Find a general formula to write $(a+ib)^{-1}$ in the form $c+id$. That is, find formulas for c and d in terms of a and b. (You can assume $a+ib$ is not 0.)

Problem 9. Prove that e^z is not equal to 0 for any $z \in \mathbb{C}$.

Problem 10. Find the following limits.

$$\lim_{x\to 0} \frac{\sin ax}{\sin bx} \qquad \lim_{x\to 0} \frac{\tan 2x}{\sin 3x} \qquad \lim_{x\to 0} \frac{\sin x}{\arctan x}$$

$$\lim_{x\to\infty} \frac{2^x}{3^x} \qquad \lim_{x\to 0} x^x$$

$$\lim_{x\to 1^-} x^{1/(1-x)} \qquad \lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x-1}$$

For review problems on differential equations, try the following problems from the textbook.

- Exercises 8.5: 1-5.
- Exercises 8.14: 1-14.
- Exercises 8.17: 1-11.

The answers for these problems are in the back of the book.