Math 2142 Exam 2 Review Solutions

Problem 1. Determine whether the following sequences converge or diverge. If they converge, find their limit.

$$a_n = \cos\frac{n\pi}{2}$$

The first sequence diverges because (starting with $n = 0$) the values repeat in the pattern $1, 0, -1, 0$.

$$a_n = \frac{n^2 + 3n - 2}{5n^2}$$

The second sequence converges to $1/5$. (To get this value, switch from n to x and use L'Hôpital's Rule or the fact that it is a rational function in which the degrees of the numerator and the denominator are equal.)

$$a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n}$$

To find the limit of the third sequence, rewrite it over a common denominator.

$$a_n = \frac{n^2}{n+1} - \frac{n^2+1}{n} = \frac{n^3 - (n^2+1)(n+1)}{n(n+1)} = \frac{n^2+n+1}{n^2+n}$$

From here, you can see the limit is 1.

$$a_n = 2^{1/n}$$

For the third sequence, as n approaches ∞, the value of $1/n$ goes to 0. Therefore, the limit is $2^0 = 1$.

$$a_n = n^{(-1)^n}$$

The fourth sequence diverges because the absolute values of the terms go to infinity.

$$a_n = \sqrt[n]{n}$$

For the last sequence, you can switch from n to x and use L'Hôpital's Rule.

$$\sqrt[x]{x} = x^{1/x} = e^{\ln x^{1/x}} = e^{\ln x/x}$$

By L'Hôpital's Rule, $\lim_{x\to\infty} \ln x/x = 0$ and so the limit of the sequence is $e^0 = 1$.

Problem 2(a). Let $\{a_n\}$ be a strictly decreasing sequence for which each term $a_n > 0$. Prove that $\lim_{n\to\infty} a_n \geq 0$.

Solution. There a couple of ways you might do this problem. They both start by noting that since $\{a_n\}$ is a bounded monotonic sequence, then it has to converge. So, the sequence has a limit and the only question is to show that the limit is non-negative.

One method to show that $\lim_{n\to\infty} a_n \geq 0$ is to note that since $\{a_n\}$ is a strictly decreasing sequence which is bounded below, we know it converges to $\inf A$ where A is the set of numbers in the sequence. Since 0 is a lower bound for A, we know that $\inf A \geq 0$ and therefore $\lim_{n\to\infty} a_n \geq 0$.

A second method to show $\lim_{n\to\infty} a_n \geq 0$ is by contradiction. Since we know that the sequence has a limit, we start by assuming that $\lim_{n\to\infty} a_n = c < 0$. We need to derive a contradiction. To get a contradiction, pick a value ε such that $0 < \varepsilon < |c|$, and so $c + \varepsilon < 0$ (since c is negative). Applying the definition of $\lim_{n\to\infty} = c$, there is an N such that if $n \geq N$, then $|a_n - c| < \varepsilon$. Removing the absolute value signs gives

$$-\varepsilon < a_n - c < \varepsilon$$

and so

$$c - \varepsilon < a_n < c + \varepsilon$$

But, as we noted above, $c + \varepsilon < 0$ and so if $n \geq N$, then a_n is negative! This contradicts the fact that each term in the sequence is positive.

2(b). Give a counterexample to show that you cannot in general conclude that $\lim_{n\to\infty} a_n > 0$.

Solution. Define a sequence by $a_n = 1/n$ for $n \geq 1$. The terms are positive and strictly decreasing, but $\lim_{n\to\infty} a_n = 0$.

Problem 3. Let $\sum_{k=0}^{\infty} a_k$ be a convergent series with positive terms. Prove that for every $\varepsilon > 0$, there is an N such that if $n \geq N$, then $\sum_{k=n+1}^{\infty} a_k < \varepsilon$.

Solution. Since the given series converges, let $\sum_{k=0}^{\infty} a_k = S$. By definition, we know that the limit of the partial sums is equal to S. That is,

$$\lim_{n\to\infty} \sum_{k=0}^{n} a_k = S$$

Now, fix $\varepsilon > 0$ and we will try to find an appropriate N. Applying the definition of $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = S$, there is an N such that if $n \geq N$, then $|S - \sum_{k=0}^{n} a_k| < \varepsilon$. (Notice that I have written the difference inside the absolute values in a different order than usual. However, because of the absolute value signs, it doesn't matter which order we subtract the terms in!) Removing the absolute value signs tells us that if $n \geq N$, then

$$-\varepsilon < S - \sum_{k=0}^{n} a_k < \epsilon$$

But, $S = \sum_{k=0}^{\infty} a_k$ and so $S - \sum_{k=0}^{n} a_k = \sum_{n+1}^{\infty} a_k$. Therefore, we have that if $n \geq N$, then

$$-\varepsilon < \sum_{k=n+1}^{\infty} a_k < \epsilon$$

and the right inequality is what we wanted to show.

Problem 4. Determine if the following telescoping series are convergent or divergent. If they converge, find the sum.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$$

Solution. Use partial fractions to decompose the fraction.

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

which means

$$1 = A(2n+1) + B(2n-1)$$

Solving these gives $A = 1/2$ and $B = -1/2$. (One way to see this is to plug in $n = 1/2$ and $n = -1/2$.) Therefore, the n-th term looks like

$$\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

The first few partial sums are

$$s_1 = \frac{1}{2(1)} - \frac{1}{2(3)}$$

$$s_2 = \left(\frac{1}{2(1)} - \frac{1}{2(3)}\right) + \left(\frac{1}{2(3)} - \frac{1}{2(5)}\right) = \frac{1}{2(1)} - \frac{1}{2(5)}$$

$$s_3 = \left(\frac{1}{2(1)} - \frac{1}{2(5)}\right) + \left(\frac{1}{2(5)} - \frac{1}{2(7)}\right) = \frac{1}{2(1)} - \frac{1}{2(7)}$$

From here, the pattern emerges: $s_n = 1/2 - 1/(2n+1)$. To find the value of the original series,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \lim_{n\to\infty} s_n = \frac{1}{2}$$

Before going on to the second telescoping series, you might try to prove that $s_n = 1/2 - 1/(2n+1)$ by induction on n. I just stated it above because the pattern is fairly clear from the first few examples, but it a good exercise to prove it by induction. Use the fact that $s_{n+1} = s_n + a_{n+1}$ and it should fall out relatively easily.

The second sum to consider is

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

For this sum, the partial faction decomposition is a longer. Since the denominator is a product of degree one factors, we write

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$$

and therefore,

$$2n + 1 = An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2$$

Plugging in $n = 0$ gives us that $B = 1$ and plugging in $n = -1$ gives us that $D = -1$. Therefore, we have

$$2n + 1 = A(n^3 + 2n^2 + n) + (n^2 + 2n + 1) + C(n^3 + n^2) - n^2$$

Collecting terms of the same degree gives us

$$2n + 1 = (A + C)n^3 + (2A + 2C)n^2 + (A + 2)n + 1$$

Comparing the n terms, we see that $2 = A + 2$ and hence $A = 0$. Comparing the n^3 terms, we see that $0 = A + C$ and hence $C = 0$. Therefore, our partial fraction decomposition is

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

The first few partial sums are

$$s_1 = \frac{1}{1^2} - \frac{1}{2^2}$$

$$s_2 = \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{1}{1^2} - \frac{1}{3^2}$$

$$s_3 = \left(\frac{1}{1^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{5^2}\right) = \frac{1}{1^2} - \frac{1}{5^2}$$

The general pattern emerges that $s_n = 1 - 1/(n+1)^2$. (As above, it is a good exercise to prove this formula by induction on n.) Therefore, our sum is

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = \lim_{n\to\infty} 1 - \frac{1}{(n+1)^2} = 1$$

Problem 5. Determine if the following geometric series converge or diverge. If they converge, find the sum.

$$\sum_{n=1}^{\infty} \frac{1+2^n}{3^n} = \sum_{n=1}^{\infty}(1/3)^n + \sum_{n=1}^{\infty}(2/3)^n = \left(\frac{1}{1-1/3} - 1\right) + \left(\frac{1}{1-2/3} - 1\right)$$

$$\sum_{n=0}^{\infty} \frac{1}{2^{n/2}} = \sum_{n=0}^{\infty}(1/\sqrt{2})^n = \frac{1}{1-1/\sqrt{2}}$$

$$\sum_{n=3}^{\infty} \frac{3^{n-1}}{e^n} = \frac{1}{3}\sum_{n=3}^{\infty}(3/e)^n \text{ which diverges since } 3/e > 1$$

Problem 6. Let $f(x)$ be a function which is continuous, strictly positive and strictly decreasing on the interval $[1, \infty)$ such that $\int_1^\infty f(x)\,dx$ converges. By the Integral Test, the series $\sum_{k=1}^\infty a_k$ with $a_k = f(k)$ converges, so we have

$$\sum_{k=1}^{\infty} a_k = S$$

For this problem, you will give an error estimate using the Integral Test for this series. Let $s_n = \sum_{k=1}^n a_k$ be the n-th partial sum for this series. The error in using s_n as an approximation to S is given by

$$R_n = S - s_n = \sum_{n+1}^{\infty} a_k$$

Draw a picture similar to the ones we used in the proof of the Integral Test to prove that $R_n \leq \int_n^\infty f(x)\,dx$.

I don't know how to draw this picture electronically but look at Figure 10.4 on page 397 to help you.

Problem 7(a). By the Integral Test, we know that $\sum_{k=1}^\infty 1/k^3$ converges. Use Problem 6 to give an upper bound for the error in using $\sum_1^{10} 1/k^3$ to approximate the value of this series.

7(b). What is the least value of n for which $s_n = \sum_{k=1}^n 1/k^3$ is accurate approximation to within 5×10^{-4}?

Solution. To do this problem, notice that the function we are working with is $f(x) = 1/x^3$. By Problem 6, we have that the error in using s_{10} to approximate the given series is less than $\int_{10}^\infty 1/x^3\,dx$. Therefore, we calculate

$$\int_{10}^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_{10}^{t} x^{-3}\,dx = \lim_{t\to\infty} -1/(2x^2)\big|_{10}^{t} = \lim_{t\to\infty} -1/(2t^2) + 1/200 = 1/200$$

To do 7(b), we need to find the least n such that $\int_n^\infty x^{-3}\,dx < 5 \times 10^{-4}$.

$$\int_{n}^{\infty} x^{-3}\,dx = \lim_{t\to\infty} \int_{n}^{t} x^{-3}\,dx = \lim_{t\to\infty} -1/(2x^2)\big|_{n}^{t} = \lim_{t\to\infty} -1/(2t^2) + 1/(2n^2) = 1/(2n^2)$$

Therefore, we need to find the least n such that $1/(2n^2) < 5 \times 10^{-4}$, which is the same as finding the least n such that $1/n^2 < 10^{-3}$. I'll leave you to find the exact value of n.

Problem 8. Let $f_n(x) = (\sin nx)/n$ and let $f(x) = \lim_{n\to\infty} f_n(x)$ be the pointwise limit of the sequence of functions $\{f_n(x)\}$. Show that $f(x)$ is defined for all x and that $f(x) = 0$. Then show that

$$\lim_{n\to\infty} f_n'(0) \neq f'(0)$$

This example shows that limits of sequences of functions cannot always be interchanged with derivatives.

Solution. To show that $f(x) = 0$, we use the Squeeze Theorem. For any value of x, we have

$$-\frac{1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n}$$

Since $\lim_{n\to\infty} 1/n = 0$, the Squeeze Theorem tells us that $\lim_{n\to\infty}(\sin nx)/n = 0$.

To do the second part of the problem, we have $f_n'(x) = \cos nx$ and therefore, $f_n'(0) = 1$ for each n. This tells us that $\lim_{n\to\infty} f_n'(0) = \lim_{n\to\infty} 1 = 1$. On the other hand, $f(x) = 0$, so $f'(x) = 0$ and $f'(0) = 0$. Therefore, $\lim_{n\to\infty} f_n'(0) = 1 \neq 0 = f'(0)$.

Problem 9(a). Prove the following integration formula for integers $n \geq 1$.

$$\int_0^\pi \frac{\sin nx}{n^2}\,dx = \begin{cases} 2/n^3 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Solution. We calculate the integral by

$$\int_0^\pi \frac{\sin nx}{n^2}\,dx = \frac{-\cos nx}{n^3}\bigg|_0^\pi = \frac{-\cos n\pi}{n^3} - \frac{-\cos 0}{n^3} = \frac{-\cos n\pi + 1}{n^3}$$

Suppose n is odd. In this case, $-\cos n\pi = -(-1) = 1$ and the value of the integral is $2/n^3$. On the other hand, if n is even, then $-\cos n\pi = -1$ and the value of the integral is 0.

9(b). Prove that the series $\sum_{n=1}^\infty (\sin nx)/n^2$ converges absolutely for all x. Let $f(x)$ denote the value of this sum.

Solution. To test for absolute convergence, we can take the absolute value and use the Comparison Test.

$$0 \leq \frac{|\sin nx|}{n^2} \leq \frac{1}{n^2}$$

Since the series $\sum_{n=1}^\infty 1/n^2$ converges, the Comparison Test tells us that $\sum_{n=1}^\infty (|\sin nx|)/n^2$ converges and hence the series $\sum_{n=1}^\infty (\sin nx)/n^2$ converges absolutely.

9(c). Use the Weierstrass M-Test to prove that the series of functions $\sum_{n=1}^\infty (\sin nx)/n^2$ converges uniformly to $f(x)$.

Solution. To use the M-Test, we note that $|\sin nx|/n^2 \leq 1/n^2$ for all x. Therefore, we can set $M_n = 1/n^2$. Since $\sum_{n=1}^\infty M_n = \sum_{n=1}^\infty 1/n^2$ converges, the M-Test tells us that $\sum_{n=1}^\infty (\sin nx)/n^2$ converges uniformly to its limit $f(x)$.

9(d). Explain why $f(x)$ is continuous (and hence integrable) on the interval $[0, \pi]$.

Solution. Since $\sum_{n=1}^\infty (\sin nx)/n^2$ converges uniformly to $f(x)$ and since each function $(\sin nx)/n$ is continuous, we know that the uniform limit function $f(x)$ is also continuous.

9(e). Use the uniform convergence of $\sum_{n=1}^{\infty}(\sin nx)/n^2$ to $f(x)$ to integrate term-by-term and prove the formula

$$\int_0^{\pi} f(x)\,dx = \sum_{k=1}^{\infty} \frac{2}{(2k-1)^3}$$

Solution. To see the process of taking the term-by-term integration, it might be easier to write out our function.

$$f(x) = \frac{\sin x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} + \frac{\sin 4x}{4^2} + \frac{\sin 5x}{5^2} + \cdots$$

Because of the uniform convergence, we have

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{1^2}\,dx + \int_0^{\pi} \frac{\sin 2x}{2^2}\,dx + \int_0^{\pi} \frac{\sin 3x}{3^2}\,dx + \int_0^{\pi} \frac{\sin 4x}{4^2}\,dx + \int_0^{\pi} \frac{\sin 5x}{5^2} + \cdots$$

By this first part of this problem, the terms with even n all have integrals equal to 0, so we have

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{\sin x}{1^2}\,dx + \int_0^{\pi} \frac{\sin 3x}{3^2}\,dx + \int_0^{\pi} \frac{\sin 5x}{5^2} + \cdots$$

Again, using the first part, the integrals when n is odd evaluate to $2/n^3$ so we have

$$\int_0^{\pi} f(x)\,dx = \frac{2}{1^3} + \frac{2}{3^3} + \frac{2}{5^3} + \cdots$$

We can write the sum on the right side as $\sum_{k=1}^{\infty} 2/(2k-1)^3$ using the fact that the terms $(2k-1)^3$ as k ranges from 1 to ∞ give us exactly the terms n^3 for the odd numbers n from 1 to ∞.