Solutions to Homework 9

Problem 1. Verify that \( f(x) = 2x^2 - 4x + 5 \) satisfies the hypotheses of Rolle’s Theorem on \([-1, 3]\) and find all points \( c \in [-1, 3] \) for which \( f'(c) = 0 \).

Solution. Since \( f \) is a polynomial, it is continuous and differentiable everywhere. Also, \( f(-1) = f(3) = 11 \), so it satisfies the conditions for Rolle’s Theorem. The only root of \( f'(x) = 4x - 4 \) is \( x = 1 \), so \( c = 1 \) is the only point satisfying Rolle’s Theorem in \([-1, 3]\).

Problem 2. Let \( f(x) = x^3 - 3x + 1 \). Prove there is exactly one point \( c \in [-1, 1] \) such that \( f(c) = 0 \).

Solution. The function \( f(x) \) is a polynomial, so it is continuous everywhere. Since \( f(-1) = 3 \) and \( f(1) = -1 \), the Intermediate Value Theorem says that \( f(x) \) has a root in \((-1, 1)\). Therefore, \( f(x) \) has at least one root in this interval.

To see that \( f(x) \) has at most one root in this interval, suppose for a contradiction that \( f \) had two roots, \( c_1 < c_2 \) in \((-1, 1)\). Because \( f \) is a polynomial, it is differentiable everywhere. Since \( f(c_1) = f(c_2) = 0 \), we can apply Rolle’s Theorem to get a point \( d \in (c_1, c_2) \) for which \( f'(d) = 0 \). In particular, since \( c_1, c_2 \in (-1, 1) \) and \( c_1 < d < c_2 \), we have \( d \in (-1, 1) \). However, \( f'(x) = 3x^2 - 3 \) and \( 3x^2 - 3 = 0 \) if and only if \( x = \pm 1 \). So, there cannot be a point \( d \in (-1, 1) \) for which \( f'(d) = 0 \) giving us the required contradiction.

Problem 3. Check that \( f(x) = x^2 + 4x - 1 \) satisfies the conditions of the Mean Value Theorem on the interval \([0, 2]\) and find all values \( c \) such that \( f'(c) \) is equal to the slope of the secant line connecting the points \((0, f(0))\) and \((2, f(2))\).

Solution. The function \( f \) is a polynomial, so it is continuous and differentiable everywhere. Therefore, it satisfies the hypotheses of the Mean Value Theorem. We need to find a value \( c \in (0, 2) \) such that

\[
 f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{11 - (-1)}{2} = 6.
\]

Since \( f'(x) = 2x + 4 \), we set \( 2x + 4 = 6 \) and solve to get \( x = 1 \). Therefore, \( c = 1 \) is the only point satisfying the Mean Value Theorem on \([0, 2]\).

Problem 4. Let \( f(x) = \alpha x^2 + \beta x + \gamma \) be a quadratic function where \( \alpha, \beta, \gamma \in \mathbb{R} \). Consider the closed interval \([a, b]\) with midpoint \( c = (a + b)/2 \). Prove that the slope of the secant line joining the points \((a, f(a))\) and \((b, f(b))\) is equal to the slope of the tangent line to \( f(x) \) at \( c \).

Solution. We need to show that

\[
 f' \left( \frac{a + b}{2} \right) = \frac{f(b) - f(a)}{b - a}
\]

To see these two values are equal, we calculate them separately. For the left hand side,
\(f'(x) = 2ax + \beta,\) so
\[
f'(\frac{a+b}{2}) = 2\alpha \frac{a+b}{2} + \beta = \alpha(a+b) + \beta
\]

For the right hand side,
\[
\frac{f(b) - f(a)}{b-a} = \frac{\alpha b^2 + \beta b + \gamma - (\alpha a^2 + \beta a + \gamma)}{b-a}
\]
\[
= \frac{\alpha b^2 + \beta b - \alpha a^2 - \beta a}{b-a}
\]
\[
= \frac{\alpha(b^2 - a^2) + \beta(b-a)}{b-a}
\]
\[
= \frac{\alpha(b-a)(b+a) + \beta(b-a)}{b-a}
\]
\[
= \alpha(b+a) + \beta
\]

Comparing the simplified forms of each side shows that they are equal as required.

**Problem 5.** Let \(f(x) = x^{2/3}\) and note that \(f(-1) = f(1) = 1.\) Show that there is no \(c \in [-1, 1]\) for which \(f'(c) = 0.\) Why doesn’t this contradict Rolle’s Theorem?

**Solution.** This does not contradict Rolle’s Theorem because \(f'(x) = (2/3)x^{-1/3}\) is not defined at 0. Therefore, \(f\) is not differentiable on the interval \((-1, 1)\) and hence does not satisfy the hypotheses of Rolle’s Theorem.

**Problem 6.** Let \(f(x)\) be continuous on \([a,b],\) differentiable on \((a,b)\) and satisfy \(f(x) \geq 0\) on \([a,b].\) Let \(g(x) = f(x)^2.\) Prove that for any \(c \in (a,b),\) \(f'(c) = 0\) if and only if \(g'(c) = 0.\)

**Solution.** Applying the chain rule to differentiate \(g(x) = f(x)^2,\) we have \(g'(x) = 2f(x)f'(x).\) There are two different implications to prove in this problem.

First, we prove that if \(f'(c) = 0,\) then \(g'(c) = 0.\) Assume \(f'(c) = 0.\) Substituting \(x = c\) into the formula for \(g'(x)\) yields
\[
g'(c) = 2f(c)f'(c) = 2f(c) \cdot 0 = 0
\]
Therefore, \(g'(c) = 0\) as required.

Second, we prove that if \(g'(c) = 0,\) then \(f'(c) = 0.\) Substituting \(x = c\) into the formula for \(g'(x)\) yields
\[
g'(c) = 2f(c)f'(c)
\]
Since \(g'(c) = 0,\) we know \(2f(c)f'(c) = 0,\) and so \(f(c)f'(c) = 0.\) If two real numbers multiply to give 0, one of them must be 0. Therefore, we know that either \(f'(c) = 0\) or \(f(c) = 0.\) If \(f'(c) = 0,\) then we are done (because that is what we want to show). So, we only have to consider the case when \(f(c) = 0.\) Since \(c \in (a,b)\) by hypothesis, and since \(f(x) \geq 0\) on \((a,b),\) the function \(f(x)\) must have a local minimum at \(c,\) which means that either \(f\) is not
differentiable at \( c \) or \( f'(c) = 0 \). We know \( f \) is differentiable at \( c \) because \( c \in (a, b) \), so we must have \( f'(c) = 0 \) as required.

**Problem 7.** Use the Fundamental Theorem of Calculus to find derivatives for the following functions. You don’t need to simplify your answers.

\[
\begin{align*}
f(x) &= \int_{5}^{x} \sin t^2 \, dt \\
g(x) &= \int_{5}^{x^2} \sin t \, dt \\
h(x) &= \int_{x}^{x^2+3} t^2 + 4t + 2 + \cos(t) \, dt
\end{align*}
\]

**Solution.** To find \( f'(x) \), you can use the Fundamental Theorem directly:

\[
f'(x) = \sin x^2
\]

To find \( g'(x) \), you need to use the chain rule and the Fundamental Theorem. Think of \( g(x) \) as the composition of the function \( \int_{5}^{u} \sin t \, dt \) and \( u = x^2 \). The integral is the “outside function” and \( u = x^2 \) is the “inside function”. By the chain rule

\[
g'(x) = \sin(x^2) \cdot 2x
\]

To find \( h'(x) \), you need to split up the integral to put it in a form to apply the Fundamental Theorem.

\[
h(x) = \int_{x}^{0} t^2 + 4t + 2 + \cos(t) \, dt + \int_{x}^{x^2+3} t^2 + 4t + 2 + \cos(t) \, dt
\]

\[
h(x) = -\int_{0}^{x} t^2 + 4t + 2 + \cos(t) \, dt + \int_{0}^{x^2+3} t^2 + 4t + 2 + \cos(t) \, dt
\]

Now the integrals are written so that the Fundamental Theorem can be applied. In the case of the second integral, you need to use the chain rule as in the case for \( g(x) \).

\[
h'(x) = -(x^2 + 4x + 2 + \cos(x)) + [(x^2 + 3)^2 + 4(x^2 + 3) + 2 + \cos(x^2 + 3)] \cdot 2x
\]

**Problem 8.** Let \( f(t) \) be a function which is continuous everywhere and satisfies the equation

\[
\int_{0}^{x} f(t) \, dt = -\frac{1}{2} + x^2 + x \sin(2x) + \frac{1}{2} \cos(2x)
\]

Find \( f(\pi/4) \). (Hint: Use the Fundamental Theorem of Calculus.)
**Solution.** Let \( g(x) = \int_{0}^{x} f(t) \, dt \), which means that \( g(x) = -\frac{1}{2} + x^2 + x \sin(2x) + \frac{1}{2} \cos(2x) \) as well. From the first description of \( g(x) \), we know that \( g'(x) = f(x) \) by the Fundamental Theorem. From the second description of \( g(x) \), we know that \( g'(x) = 2x + \sin(2x) + 2x \cos(2x) - \sin(2x) \). Therefore,

\[
f(x) = 2x + \sin(2x) + 2x \cos(2x) - \sin(2x)
\]

and so

\[
f(\pi/4) = \pi/2 + \sin(\pi/2) + 2(\pi/4) \cos(\pi/2) - \sin(\pi/2) = \pi/2
\]

**Problem 9.** Find the following indefinite integrals. You may find it helpful to do some simplifying before calculating the integral.

\[
\int (x + 1)(x^3 - 2) \, dx
\]

\[
\int \sqrt{2x} + \sqrt{x/2} \, dx
\]

\[
\int \frac{x^4 + x - 3}{x^3} \, dx
\]

**Solution.** For the first integral

\[
\int (x + 1)(x^3 - 2) \, dx = \int x^4 + x^3 - 2x - 2 \, dx = x^5/5 + x^4/4 - x^2 - 2x + c
\]

For the second integral

\[
\int \sqrt{2x} + \sqrt{x/2} \, dx = \int \sqrt{2} \cdot x^{1/2} + x^{1/2} / \sqrt{2} \, dx = (\sqrt{2} + 1/\sqrt{2}) \int x^{1/2} \, dx = (\sqrt{2} + 1/\sqrt{2}) \cdot (2/3) x^{3/2} + c
\]

For the third integral

\[
\int \frac{x^4 + x - 3}{x^3} \, dx = \int x + x^{-2} - 3x^{-3} \, dx = x^2/2 - x^{-1} + (3/2)x^{-2} + c
\]