Math 2141: Homework 9, Due Friday November 8

Problem 1. Verify that \( f(x) = 2x^2 - 4x + 5 \) satisfies the hypotheses of Rolle’s Theorem on \([-1, 3]\) and find all points \( c \in [-1, 3] \) for which \( f'(c) = 0 \).

Problem 2. Let \( f(x) = x^3 - 3x + 1 \). Prove there is exactly one point \( c \in [-1, 1] \) such that \( f(c) = 0 \).

Hint. First, you need to explain why \( f(x) = 0 \) has at least one solution in \([a, b]\). Second, you need to prove that \( f(x) = 0 \) cannot have two solutions in \([a, b]\). I would do this part by contradiction. Assume that there are \( c_1 \neq c_2 \) in \([a, b]\) such that \( f(c_1) = f(c_2) = 0 \) and derive a contradiction.

Problem 3. Check that \( f(x) = x^2 + 4x - 1 \) satisfies the conditions of the Mean Value Theorem on the interval \([0, 2]\) and find all values \( c \) such that \( f'(c) \) is equal to the slope of the secant line connecting the points \((0, f(0))\) and \((2, f(2))\).

In Problem 3, you should have found a single value for \( c \) and it should be the midpoint of the interval \([0, 2]\). In the next problem, you will show that as long as \( f(x) \) is a quadratic function, the value \( c \) in the Mean Value Theorem applied to \( f(x) \) on \([a, b]\) is always the midpoint of \([a, b]\). This is a special property of quadratic functions!

Problem 4. Let \( f(x) = \alpha x^2 + \beta x + \gamma \) be a quadratic function where \( \alpha, \beta, \gamma \in \mathbb{R} \). Consider the closed interval \([a, b]\) with midpoint \( c = (a + b)/2 \). Prove that the slope of the secant line joining the points \((a, f(a))\) and \((b, f(b))\) is equal to the slope of the tangent line to \( f(x) \) at \( c \).

Problem 5. Let \( f(x) = x^{2/3} \) and note that \( f(-1) = f(1) = 1 \). Show that there is no \( c \in [-1, 1] \) for which \( f'(c) = 0 \). Why doesn’t this contradict Rolle’s Theorem?

Problem 6. Let \( f(x) \) be continuous on \([a, b]\), differentiable on \((a, b)\) and satisfy \( f(x) \geq 0 \) on \([a, b]\). Let \( g(x) = f(x)^2 \). Prove that for any \( c \in (a, b) \), \( f'(c) = 0 \) if and only if \( g'(c) = 0 \).

Hint. The first thing I would do is use the chain rule to get a formula for \( g'(x) \). Once you have that, you need to show two things in this problem. First, assume that \( f'(c) = 0 \), and show that \( g'(c) = 0 \). This implication follows almost immediately. Second, assume that \( g'(c) = 0 \), and show that \( f'(c) = 0 \). This implication takes a bit more thought. Make sure you explain your reasoning very clearly.
Problem 7. Use the Fundamental Theorem of Calculus to find derivatives for the following functions. You don’t need to simplify your answers.

\[ f(x) = \int_{5}^{x} \sin t^2 \, dt \]
\[ g(x) = \int_{5}^{x^2} \sin t \, dt \]
\[ h(x) = \int_{x}^{x^2+3} t^2 + 4t + 2 + \cos(t) \, dt \]

Problem 8. Let \( f(t) \) be a function which is continuous everywhere and satisfies the equation

\[ \int_{0}^{x} f(t) \, dt = -\frac{1}{2} + x^2 + x \sin(2x) + \frac{1}{2} \cos(2x) \]

Find \( f(\pi/4) \). (Hint: Use the Fundamental Theorem of Calculus.)

Problem 9. Find the following indefinite integrals. You may find it helpful to do some simplifying before calculating the integral.

\[ \int (x + 1)(x^3 - 2) \, dx \]
\[ \int \sqrt{2x} + \sqrt{x/2} \, dx \]
\[ \int \frac{x^4 + x - 3}{x^3} \, dx \]