Math 2141: Homework 6 Solutions

Problem 1. Give an $\varepsilon - \delta$ proof that $\lim_{x \to 1} 2x + 3 = 5$.

Solution. We are given $\varepsilon > 0$ and need to find $\delta > 0$ such that if $0 < |x - 1| < \delta$, then $|(2x + 3) - 5| < \varepsilon$.

(Side calculation: We simplify what we need to show to find an appropriate $\delta$.

\[ |(2x + 3) - 5| < \varepsilon \iff |2x - 2| < \varepsilon \iff |x - 1| < \varepsilon/2 \]

This completes the side calculation. We return to the actual proof.)

Let $\delta = \varepsilon/2$. To show that this value of $\delta$ works, we assume that $0 < |x - 1| < \varepsilon/2$ and we show that $|(2x + 3) - 5| < \varepsilon$.

\[ |x - 1| < \varepsilon/2 \Rightarrow 2|x - 1| < \varepsilon \Rightarrow |2x - 2| < \varepsilon \Rightarrow |(2x + 3) - 5| < \varepsilon \]

as required.

Problem 2. Give an $\varepsilon - \delta$ proof that $\lim_{x \to p} -3x + 2 = -3p + 2$.

Solution. We are given $\varepsilon > 0$ and need to find $\delta > 0$ such that if $0 < |x - p| < \delta$, then $|(-3x + 2) - (-3p + 2)| < \varepsilon$.

(Side calculation. We simplify what we need to show to find an appropriate $\delta$.

\[ |(-3x + 2) - (-3p + 2)| < \varepsilon \iff |-3x + 2 + 3p - 2| < \varepsilon \]
\[ \iff |-3x + 3p| < \varepsilon \]
\[ \iff |-3(x - p)| < \varepsilon \]
\[ \iff |3| \cdot |(x - p)| < \varepsilon \]
\[ \iff |x - p| < \varepsilon/3 \]

Therefore, we want $\delta = \varepsilon/3$. This completes the side calculation. We return to the proof.)

Let $\delta = \varepsilon/3$. To show this value of $\delta$ works, we assume $0 < |x - p| < \varepsilon/3$ and we show that $|(-3x + 2) - (-3p + 2)| < \varepsilon$.

\[ |x - p| < \varepsilon/3 \Rightarrow |3| \cdot |(x - p)| < \varepsilon \]
\[ \Rightarrow |-3(x - p)| < \varepsilon \]
\[ \Rightarrow |-3x + 3p| < \varepsilon \]
\[ \Rightarrow |-3x + 2 + 3p - 2| < \varepsilon \]
\[ \Rightarrow |(-3x + 2) - (-3p + 2)| < \varepsilon \]

as required.
Problem 3. Let \( f(x) = ax + b \) where \( a \) and \( b \) are fixed real numbers. Given an \( \varepsilon - \delta \) proof that \( \lim_{x \to p} f(x) = ap + b \). (Be careful because \( a \) could be negative!)

Solution. First, consider the case when \( a = 0 \). If \( a = 0 \), then \( f(x) = b \) is a constant function and
\[
\lim_{x \to p} f(x) = \lim_{x \to p} b = b = f(p).
\]
This proves the case when \( a = 0 \). From now on, we can assume that \( a \neq 0 \).

Assume that \( a \neq 0 \). We are given \( \varepsilon > 0 \) and need to find \( \delta > 0 \) such that if \( 0 < |x - p| < \delta \), then \(|(ax - b) - (ap - b)| < \varepsilon\).

(Side calculation: We simplify what we need to show to find an appropriate \( \delta \).
\[
|\(ax + b\) - (ap + b)| < \varepsilon \iff |ax - ap| < \varepsilon \iff |a| |x - p| < \varepsilon \iff |x - p| < \varepsilon/|a|
\]
This completes the side calculation. We return to the actual proof.)

Let \( \delta = \varepsilon/|a| \). To show that value of \( \delta \) works, we assume that \( |x - p| < \varepsilon/|a| \) and show that \(|(ax + b) - (ap + b)| < \varepsilon\).
\[
|x - p| < \varepsilon/|a| \Rightarrow |a| |x - p| < \varepsilon \Rightarrow |ax - ap| < \varepsilon \Rightarrow |(ax + b) - (ap + b)| < \varepsilon
\]
as required.

Problem 4(a). Prove that if \( |x - 1| < 1 \), then \( |x^2 - 1| < 3|x - 1| \).

Solution. To prove this statement, we assume that \( x \) is a number such that \( |x - 1| < 1 \). We have to show that \( |x^2 - 1| < 3|x - 1| \).

Removing the absolute value signs from \( |x - 1| < 1 \) tells us that \(-1 < x - 1 < 1\). Adding 1 to both sides gives us \( 0 < x < 2 \). Therefore, we know that \( 1 < x + 1 < 3 \).

Now, work on \( |x^2 - 1| \) for a moment. We can factor \( x^2 - 1 = (x + 1)(x - 1) \). Therefore
\[
|x^2 - 1| = |(x + 1)(x - 1)| = |x + 1| \cdot |x - 1|
\]
Since we know that the value of \( x \) satisfies \( 1 < x + 1 < 3 \), we can remove the absolute value signs on \( |x + 1| \) and then use the fact that \( x + 1 < 3 \):
\[
|x^2 - 1| = (x + 1) \cdot |x - 1| < 3|x - 1|
\]
Therefore \( |x^2 - 1| < 3|x - 1| \) as required.

4(b). Give an \( \varepsilon - \delta \) proof that \( \lim_{x \to 1} x^2 = 1 \).

Solution. We are given \( \varepsilon > 0 \) and have to find \( \delta > 0 \) such that if \( 0 < |x - 1| < \delta \) then \( |x^2 - 1| < \varepsilon \).

(Side calculation: We simplify what we need to show to find an appropriate \( \delta \). By 4(a), we know that when \( |x - 1| < 1 \), we have that \( |x^2 - 1| < 3|x - 1| \). Therefore, if we want \( |x^2 - 1| < \varepsilon \), then as long as we guarantee that \( \delta < 1 \), it suffices to make sure that \( 3|x - 1| < \varepsilon \). Or, in
other words, that \(|x - 1| < \varepsilon/3\). Therefore, we want to use \(\delta = \varepsilon/3\). This completes the side calculation. We return to the proof.)

Let \(\delta\) be the minimum of 1 and \(\varepsilon/3\). To show this \(\delta\) works, we assume that \(0 < |x - 1| < \delta\) and show that \(|x^2 - 1| < \varepsilon\). Since \(\delta \leq 1\), we know that \(|x^2 - 1| < 3|x - 1|\) by 7(a). Since \(\delta \leq \varepsilon/3\), we know that \(|x - 1| < \varepsilon/3\). Therefore, we have

\[
|x - 1| < \delta \quad \Rightarrow \quad |x - 1| < \varepsilon/3
\]
\[
\Rightarrow \quad 3|x - 1| < \varepsilon
\]
\[
\Rightarrow \quad |x^2 - 1| < 3|x - 1| < \varepsilon
\]

and so \(|x^2 - 1| < \varepsilon\) as required.

**Problem 5.** Suppose that \(\lim_{x \to p^+} f(x) = r\) and \(r > 0\). Prove that there is a neighborhood \(N(p)\) of \(p\) such that for all \(x \in N(p)\) with \(x > p\), we have \(f(x) > 0\).

**Solution.** Since we know that \(\lim_{x \to p} f(x) = r\), we can specify a value of \(\varepsilon\) (as long as it is positive) and we get back an appropriate value of \(\delta\). Let \(\varepsilon = r/2\) and note that \(\varepsilon > 0\).

Because \(\lim_{x \to p} f(x) = r\), we know there is a value \(\delta > 0\) such that if \(0 < |x - p| < \delta\), then \(|f(x) - r| < r/2\). Let \(N(p)\) be the neighborhood of \(p\) with radius \(\delta\). To check that this neighborhood works, we need to show that if \(x \in N(p)\) and \(x \neq p\), then \(f(x) > 0\). Therefore, we assume \(x \in N(p)\) with \(x \neq p\) and we show that \(f(x) > 0\).

By the definition of \(N(p)\), this means the distance from \(x\) to \(p\) is strictly less than \(\delta\), or in other words, \(|x - p| < \delta\). Since \(x \neq p\), we know \(0 < |x - p|\) and so \(0 < |x - p| < \delta\).

Since \(0 < |x - p| < \delta\) implies \(|f(x) - r| < r/2\), we conclude that \(|f(x) - r| < r/2\). Removing the absolute value signs tells us that \(-r/2 < f(x) - r < r/2\). Adding \(r\) to both sides gives \(r/2 < f(x) < 3r/2\). Since \(0 < r/2\), we have established that \(0 < f(x)\) which is what we wanted to show.

**Problem 6.** Define a function \(f(x)\) as follows.

\[
f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 0 \\
  2x - 3 & \text{if } x > 0 
\end{cases}
\]

Does \(\lim_{x \to 0} f(x)\) exist? If so, what it is? If not, why not?

**Solution.** Consider the two one sided limits: \(\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2x - 3 = -3\) and \(\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x + 1 = 1\). Since the one-sided limits are not equal, \(\lim_{x \to 0} f(x)\) does not exist.

**Problem 7.** Let \(a, b\) and \(c\) be fixed real numbers. Define a function \(f(x)\) by

\[
f(x) = \begin{cases} 
  -x + 2 & \text{if } x \leq 0 \\
  ax^2 + bx + c & \text{if } x > 0 
\end{cases}
\]
For which values of $a$, $b$ and $c$ is $f(x)$ continuous at 0?

**Solution.** We need to determine for which values of $a$, $b$ and $c$, we have $\lim_{x \to 0} f(x) = f(0)$. Plugging in $x = 0$, we have $f(0) = 2$. Taking the one sided limits gives $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x + 2 = 2$, and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} ax^2 + bx + c = c$. In order to have $\lim_{x \to 0} f(x) = 2$, we need $c = 2$ and the values of $a$ and $b$ can be arbitrary.