Math 2141 Homework 3 Solutions

Problem 1. Prove that the inequality

$$||x| - |y|| \leq |x - y|$$

holds true for all real numbers x and y.

Solution: We begin by rewriting what we want to show. Using Theorem I.38 to remove the outer absolute value signs, we have

$$||x| - |y|| \leq |x - y| \Leftrightarrow -|x - y| \leq |x| - |y| \leq |x - y|$$

Therefore, we can solve the problem by showing that $|x| - |y| \leq |x - y|$ and $-|x - y| \leq |x| - |y|$. Consider the first inequality

$$|x| - |y| \leq |x - y| \qquad \textbf{(to show)}$$

This inequality is equivalent to $|x| \leq |y| + |x - y|$. So, we have turned our problem into

$$|x| \leq |y| + |x - y| \qquad \textbf{(to show)}$$

By the Triangle Inequality, we know $|u + v| \leq |u| + |v|$ for all real numbers u and v. Let $u = y$ and $v = x - y$. Plugging into the Triangle Inequality gives

$$|u + v| = |y + (x - y)| = |x| \qquad \text{and} \qquad |u| + |v| = |y| + |x - y|$$

Therefore, by the Triangle Inequality, $|x| \leq |y| + |x - y|$, which is what we needed to show. Now, we consider the second inequality above:

$$-|x - y| \leq |x| - |y| \qquad \textbf{(to show)}$$

Rearranging the text algebraically, this inequality is the same as

$$|y| \leq |x| + |x - y| \qquad \textbf{(to show)}$$

Also, since $y - x = -(x - y)$, we have $|x - y| = |y - x|$. So, we can rewrite one more time

$$|y| \leq |x| + |y - x| \qquad \textbf{(to show)}$$

Now we are in a position to use the Triangle Inequality just as before. Let $u = x$ and $v = y - x$, so

$$|u + v| = |x + (y - x)| = |y| \qquad \text{and} \qquad |u| + |v| = |x| + |y - x|$$

By the Triangle Inequality, we have $|y| \leq |x| + |y - x|$, which is exactly what we needed to show.

Problem 2. Let $p, n \in \mathbb{N}^+$. Prove that

$$n^p < \frac{(n+1)^{p+1} - n^{p+1}}{p+1} < (n+1)^p$$

Solution: Multiplying the inequalities that we need to prove by $p+1$, we can reduce the problem to proving that

$$(p+1)n^p < (n+1)^{p+1} - n^{p+1} < (p+1)(n+1)^p \qquad \textbf{(to show)}$$

First, consider the middle term. The Difference of Powers Formulas is

$$a^{p+1} - b^{p+1} = (a-b)\big(a^p + a^{p-1}b + a^{p-2}b^2 + \cdots + ab^{p-1} + b^p\big) = (a-b)\sum_{i=0}^{p} a^{p-i}b^i$$

Letting $a = n+1$ and $b = n$, we have

$$(n+1)^{p+1} - n^{p+1} = ((n+1) - n)\sum_{i=0}^{p}(n+1)^{p-i}n^i = \sum_{i=0}^{p}(n+1)^{p-i}n^i$$

Notice that since i goes from 0 to p, there are $p+1$ many terms in this sum.

Now consider the two outside terms. The term $(p+1)n^p$ is the same as adding n^p to itself $p+1$ many times. So, we can write $(p+1)n^p = \sum_{i=0}^{p} n^p$. Similarly, $(p+1)(n+1)^p = \sum_{i=0}^{p}(n+1)^p$.

We can now rewrite what we need to show as follows:

$$\sum_{i=0}^{p} n^p < \sum_{i=0}^{p}(n+1)^{p-i}n^i < \sum_{i=0}^{p}(n+1)^p \qquad \textbf{(to show)}$$

There are exactly $p+1$ terms in each of these sums, so we can compare them term by term. That is, we need to compare n^p, $(n+1)^{p-i}n^i$ and $(n+1)^p$ for each number i between 0 and p.

Consider n^p, $(n+1)^{p-i}n^i$ and $(n+1)^p$. We know $n^{p-i} \le (n+1)^{p-i}$ (and this inequality is strict unless $i = p$). Multiplying by n^i gives $n^{p-i}n^i \le (n+1)^{p-i}n^i$, which means

$$n^p \le (n+1)^{p-i}n^i$$

(and this is strict unless $i = p$). On the other hand, $n^i \le (n+1)^i$ (and this inequality is strict unless $i = 0$). Multiplying by $(n+1)^{p-i}$ gives $(n+1)^{p-i}n^i \le (n+1)^{p-i}(n+1)^i$ which means

$$(n+1)^{p-i}n^i \le (n+1)^p$$

(and this is strict unless $i = 0$). Therefore, we have shown that for each i, we have

$$n^p \le (n+1)^{p-i}n^i \le (n+1)^p$$

The left inequality is strict unless $i = p$ and the right inequality is strict unless $i = 0$. Since $p \geq 1$, at least one of these inequalities is always strict. Taking sums, we get

$$\sum_{i=0}^{p} n^p < \sum_{i=0}^{p} (n+1)^{p-i} n^i < \sum_{i=0}^{p} (n+1)^p$$

which is exactly what we needed to show.

Problem 3. Prove by induction on n that

$$\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$$

Solution: We proceed by induction on n. For the induction case, I will handle the two inequalities separately. You don't have to do it this way, but it might make the proof easier to understand.

Base case (n=1): When $n = 1$, the lefthand sum is $\sum_{k=1}^{0} k^p$ which by definition is equal to 0 because the lower index is strictly greater than the upper index. The middle term is $\frac{1^{p+1}}{p+1}$ which is equal to $\frac{1}{p+1}$. The righthand sum is $\sum_{k=1}^{1} k^p$ which is just 1. Since $0 < \frac{1}{p+1} < 1$, we have established the base case for both inequalities.

Induction case for left inequality: We first set up the induction hypothesis.

> **Assume:** For a fixed n, $\sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p+1}$
>
> **Show:** $\sum_{k=1}^{n} k^p < \frac{(n+1)^{p+1}}{p+1}$

Separating off the last term of $\sum_{k=1}^{n} k^p$ and applying the induction hypothesis, we get

$$\sum_{k=1}^{n} k^p = \left(\sum_{k=1}^{n-1} k^p \right) + n^p < \frac{n^{p+1}}{p+1} + n^p$$

and so we know

$$\sum_{k=1}^{n} k^p < \frac{n^{p+1}}{p+1} + n^p$$

Therefore, to complete the induction case, it suffices to show that

$$\frac{n^{p+1}}{p+1} + n^p \leq \frac{(n+1)^{p+1}}{p+1} \qquad \textbf{(to show)}$$

Multiplying by $p+1$, it suffices to show

$$n^{p+1} + (p+1)n^p \leq (n+1)^{p+1} \qquad \textbf{(to show)}$$

Finally, moving the n^{p+1} to the other side, it suffices to show

$$(p+1)n^p \leq (n+1)^{p+1} - n^{p+1} \qquad \textbf{(to show)}$$

However, that is exactly what we proved in Problem 2! Therefore, we are done with the induction case for the left inequality.

Induction case for right inequality: We first set up the induction hypothesis.

> **Assume:** For a fixed n, $\frac{n^{p+1}}{p+1} < \sum_{k=1}^{n} k^p$.
>
> **Show:** $\frac{(n+1)^{p+1}}{p+1} < \sum_{k=1}^{n+1} k^p$.

Separating off the last term of $\sum_{k=1}^{n+1} k^p$ and applying the induction hypothesis, we get

$$\sum_{k=1}^{n+1} k^p = \left(\sum_{k=1}^{n} k^p\right) + (n+1)^p > \frac{n^{p+1}}{p+1} + (n+1)^p$$

and so we know

$$\sum_{k=1}^{n+1} k^p > \frac{n^{p+1}}{p+1} + (n+1)^p$$

Therefore, to complete the induction case, it suffices to show that

$$\frac{n^{p+1}}{p+1} + (n+1)^p \geq \frac{(n+1)^{p+1}}{p+1} \qquad \textbf{(to show)}$$

Multiplying by $p+1$, it suffices to show

$$n^{p+1} + (p+1)(n+1)^p \geq (n+1)^{p+1} \qquad \textbf{(to show)}$$

Finally, moving the n^{p+1} to the other side, it suffices to show

$$(p+1)(n+1)^p \geq (n+1)^{p+1} - n^{p+1} \qquad \textbf{(to show)}$$

Again, this is exactly what we proved in Problem 2. Therefore, we are done with the induction case for the right inequality as well.

Problem 4. Recall that $[x]$ denotes the greatest integer function. For this problem, I want you to work with the function $f(x) = [2x] + 1$. So, for example, $f(2/3) = [4/3] + 1 = 2$. Sketch a graph of $f(x)$ on the interval $[0, 3]$ and calculate

$$\int_0^3 f(x)\, dx$$

Solution: The function $f(x)$ has constant value n on each open interval of the form $(\frac{n}{2}, \frac{n+1}{2})$. Therefore, we can break up the integral as follows:

$$\int_0^3 f(x)\,dx = \int_0^{1/2} 1\,dx + \int_{1/2}^1 2\,dx + \int_1^{3/2} 3\,dx + \int_{3/2}^2 4\,dx + \int_2^{5/2} 5\,dx + \int_{5/2}^3 6\,dx$$

$$\int_0^3 f(x)\,dx = 1\cdot(1/2-0) + 2\cdot(1-1/2) + 3\cdot(3/2-1) + 4\cdot(2-3/2) + 5\cdot(5/2-2) + 6\cdot(6-5/2)$$

$$\int_0^3 f(x)\,dx = 1/2 + 1 + 3/2 + 2 + 5/2 + 3 = 21/2$$

(You should also draw a sketch of the function, but the calculation above shows how to find the integral without sketching the graph.)

Problem 5. Let $g(x) = 2[x] - 1$. Sketch a graph of $g(x)$ on the interval $[0, 3]$ and calculate

$$\int_0^3 g(x)\,dx$$

Solution: As above, you should sketch the graph of the function. To calculate the integral:

$$\begin{aligned}
\int_0^3 2\,[x] - 1\,dx &= 2\int_0^3 [x]\,dx - \int_0^3 1\,dx \\
&= 2\Big(\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx\Big) - 1\cdot(3-0) \\
&= 2\big(0\cdot(1-0) + 1\cdot(2-1) + 2\cdot(3-2)\big) - 3 \\
&= 2\big(0+1+2\big) - 3 = 6 - 3 = 3
\end{aligned}$$

Problem 6. Prove Theorem 1.8 (expansion or contraction of the interval of integration). That is, let $s(x)$ be a step function on $[a, b]$. Prove that for any $k > 0$,

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)\,dx = k\int_a^b s(x)\,dx$$

Solution: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ such that $s(x)$ has constant value s_i on the i-th open subinterval. Then $\widehat{P} = \{kx_0, kx_1, \ldots, kx_n\}$ is a partition of $[ka, kb]$ on which the step function $s(x/k)$ has constant value s_i on the i-th open subinterval. Writing the integrals in the problem out as sums using these partitions, we have

$$k\int_a^b s(x)\,dx = k\sum_{i=1}^n s_i\cdot(x_i - x_{i-1})$$

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)\,dx = \sum_{i=1}^n s_i\cdot(kx_i - kx_{i-1}) = \sum_{i=1}^n s_i\cdot k\cdot(x_i - x_{i-1})$$

We proved that multiplicative constants can be pulled outside of finite sums. Therefore,

$$k\sum_{i=1}^{n} s_i \cdot (x_i - x_{i-1}) = \sum_{i=1}^{n} s_i \cdot k \cdot (x_i - x_{i-1})$$

which proves that

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right)\,dx = k\int_{a}^{b} s(x)\,dx.$$