Math 2141 Homework 3: Due Friday September 20

Problem 1. Prove that the inequality

\[ ||x| - |y|| \leq |x - y| \]

holds true for all real numbers \( x \) and \( y \).

*Hint:* Use Theorem I.38 for the outer absolute value on the left and then try to figure out how to use Theorem I.39 (= the triangle inequality) to complete the proof.

Problem 2. Let \( p, n \in \mathbb{N}^+ \). Prove that

\[ n^p < \frac{(n + 1)^{p+1} - n^{p+1}}{p + 1} < (n + 1)^p \]

*Hint:* You might be tempted to try proving this by induction, but there is a better way. First, notice that it is equivalent to prove

\[ (p + 1)n^p < (n + 1)^{p+1} - n^{p+1} < (p + 1)(n + 1)^p \]

Try applying the Difference of Powers Formula from class. Write down this formula for the powers \( a^{p+1} - b^{p+1} \). Then plug in \( a = n + 1 \) and \( b = n \). What is \( a - b \)? Count the number of terms remaining in the formula and think about the fact that \( n^k < (n + 1)^k \) for any power \( k \).

Problem 3. Prove by induction on \( n \) that

\[ \sum_{k=1}^{n-1} k^p < \frac{n^{p+1}}{p + 1} < \sum_{k=1}^{n} k^p \]

*Hint:* Consider the two inequalities separately. That is, think of this as two separate problems and prove both inequalities by induction on \( n \). Use Problem 2 to help you do the induction cases. Once you apply the inductive hypothesis, write down what you want to show and you will see why Problem 2 is helpful.

Problem 4. Recall that \([x]\) denotes the greatest integer function. For this problem, I want you to work with the function \( f(x) = [2x] + 1 \). So, for example, \( f(2/3) = [4/3] + 1 = 2 \). Sketch a graph of \( f(x) \) on the interval \([0, 3]\) and calculate

\[ \int_0^3 f(x) \, dx \]

Problem 5. Let \( g(x) = 2[x] - 1 \). Sketch a graph of \( g(x) \) on the interval \([0, 3]\) and calculate

\[ \int_0^3 g(x) \, dx \]
Problem 6. Prove Theorem 1.8 (expansion or contraction of the interval of integration). That is, let \( s(x) \) be a step function on \([a, b]\). Prove that for any \( k > 0 \),

\[
\int_{ka}^{kb} s\left(\frac{x}{k}\right) \, dx = k \int_{a}^{b} s(x) \, dx
\]

*Hint:* Fix a partition such that \( s(x) \) is constant on each subinterval and let \( s_i \) be the value of \( s(x) \) on the \( i \)-th subinterval. Then write down the corresponding partition for \( s(x/k) \) and use the definition of the integral of a step function.